Ultra-log-concavity of domination sequences of powers of paths and cycles

Dev D. Goyal

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Abstract

A dominating set of a graph covers every vertex through membership or adjacency. Let $d_k(G)$ count the dominating sets of size k in an n-vertex graph. We prove that the normalized sequence $d_k(G)/\binom{n}{k}$ is log-concave for every power of a path or cycle. The result holds for all $n \geq 0$ and all powers $\ell \geq 1$. It establishes Conjecture 1.8 of Galvin and Zhang [1], who proved unimodality for both families. The proof encodes dominating sets by positive gaps with upper bounds. A finite Toeplitz-matrix argument shows how these gaps change when the number of selected vertices increases. It follows that the probability of deleting a selected vertex while retaining domination is nondecreasing. A separate coefficient comparison accounts for the endpoints of a path. A Lean 4 formalization proves the theorem, including the correspondence between the gap counts and actual graph-theoretic dominating sets.

1 Statement and context

All graphs are finite and simple. A set $D \subseteq V(G)$ is *dominating* if every vertex belongs to D or has a neighbor in D. Write

$$d_k(G) = \#\{D \subseteq V(G) : |D| = k,\ D \text{ is dominating}\}.$$

The ℓth power G^ℓ joins distinct vertices whose graph distance in G is at most ℓ. The path and cycle with n vertices are denoted by P_n and C_n. The conventions are that P_0, C_0 are empty graphs, C_1 is an isolated vertex, and C_2 is the complete graph on two vertices.

A nonnegative sequence $(d_0, \ldots, d_n)$ is *ultra-log-concave*, with ambient parameter n, when $p_k = d_k/\binom{n}{k}$ satisfies $p_{k-1}p_{k+1} \leq p_k^2$ for $1 \leq k < n$. In the present setting the positive support is an interval: every superset of a dominating set is dominating. Thus there is no issue concerning internal zeros.

Theorem 1.1. *For every $n \geq 0$ and $\ell \geq 1$, the domination sequences of P_n^ℓ and C_n^ℓ are ultra-log-concave. Explicitly, for either graph G and every integer $1 \leq k < n$,*

$$(k+1)(n-k+1)d_{k-1}(G)d_{k+1}(G) \ \leq \ k(n-k)d_k(G)^2. \tag{1}$$

The equivalence of (1) with the normalized inequality follows from

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)}.$$

Both formulations are proved in the accompanying Lean development.

Galvin and Zhang proved unimodality for both families and formulated the stronger statement above as Conjecture 1.8 [1]. Zhang's dissertation repeats it as Conjecture 4.1.8 on printed page 57 [2]. The result here strengthens their unimodality theorem. Its proof uses deletion probabilities to obtain the normalized adjacent inequalities (1).

The closure of log-concavity under convolution, and its Toeplitz-matrix interpretation, are classical. In particular, the convolution closure associated with Hoggar's theorem is discussed in [3, 4]. No new general convolution theorem is claimed. Section 2 supplies the finite matrix argument required here, so that all ingredients of the proof can be checked directly.

2 Finite Toeplitz positivity and conditioning

A sequence in this section is a finitely supported function $q : \mathbb{Z} \to \mathbb{R}_{\geq 0}$. Extend all coefficient sequences by zero outside their support, and write

$$(q * r)(v) = \sum_{h \in \mathbb{Z}} q(v-h)r(h).$$

All sums below are finite. Say that q has property TP_2 if

$$q(i_1 - j_1)q(i_2 - j_2) \;\geq\; q(i_1 - j_2)q(i_2 - j_1) \quad (i_1 \leq i_2,\; j_1 \leq j_2). \tag{2}$$

This says that the minors of order two of its Toeplitz matrix $T_q(i,j) = q(i-j)$ are nonnegative.

Lemma 2.1. *Indicators of integer intervals have property* TP_2*, and convolution preserves this property. Consequently the coefficient sequence of any finite product of polynomials*

$$B_b(z) = z + z^2 + \cdots + z^b \qquad (b \geq 1) \tag{3}$$

has property TP_2*. The empty product, whose sequence is* $\mathbf{1}_{\{0\}}$*, also has property* TP_2*.*

Proof. For an interval indicator, the right side of (2) is either zero or one. If it is one, both $i_1 - j_2$ and $i_2 - j_1$ belong to the interval. Each of $i_1 - j_1, i_2 - j_2$ lies between those two integers, so the left side is also one.

Suppose q and r have property TP_2. For the two fixed row indices i_1, i_2 and column indices j_1, j_2, choose a finite integer interval S containing every u for which $q(i_a - u) \neq 0$ for some $a \in \{1, 2\}$. Let A be the $2 \times |S|$ matrix with entries $A_{a,u} = q(i_a - u)$ and let B be the $|S| \times 2$ matrix with entries $B_{u,b} = r(u - j_b)$. Then $(AB)_{a,b} = (q * r)(i_a - j_b)$. Expanding the determinant and grouping the terms indexed by (u, v) and (v, u) gives the finite identity

$$\det(AB) = \sum_{\substack{u,v \in S \\ u<v}} \det \begin{pmatrix} A_{1,u} & A_{1,v} \\ A_{2,u} & A_{2,v} \end{pmatrix} \det \begin{pmatrix} B_{u,1} & B_{u,2} \\ B_{v,1} & B_{v,2} \end{pmatrix}. \tag{4}$$

Every term is nonnegative by (2), proving the claim. Applying this repeatedly to the interval indicators in (3) gives the last assertion. This is the order-two Cauchy–Binet argument; no infinite matrix product or limiting argument is needed. □

Lemma 2.2 (Positive-shift comparison)**.** *Suppose* q *has property* TP_2*, and* p *is nonnegative and supported on positive integers. Put* $q^+ = q * p$*. For* $u \leq v$*,*

$$q(u)q^+(v) \geq q^+(u)q(v). \tag{5}$$

Proof. For every h in the support of p, use rows u, v and columns $0, h$ in (2). This gives $q(u)q(v-h) \geq q(u-h)q(v)$. Multiplication by $p(h)$ and summation over h give (5). No individual coefficient is divided by. □

Lemma 2.3 (Conditional lower tails)**.** *Under the hypotheses of Lemma 2.2, fix a nonnegative finitely supported sequence w, an integer total N, and a threshold L. Define*

$$f(s) = w(s)q(N-s), \qquad f^+(s) = w(s)q^+(N-s),$$

and let $Z = \sum_s f(s)$ and $Z^+ = \sum_s f^+(s)$. Then

$$\left(\sum_{s \leq L} f(s)\right) Z^+ \;\leq\; \left(\sum_{s \leq L} f^+(s)\right) Z. \tag{6}$$

When $Z, Z^+ > 0$, the conditional probability of $s \leq L$ therefore increases after replacing q by $q * p$.

Proof. If $s < t$, then $N - t < N - s$. Lemma 2.2, multiplied by $w(s)w(t)$, gives

$$f^+(s)f(t) \geq f(s)f^+(t).$$

The difference between the right and left sides of (6) is exactly

$$\sum_{s \leq L < t} (f^+(s)f(t) - f(s)f^+(t)). \tag{7}$$

Indeed, the terms with both indices at most L cancel after exchanging the two indices. Every remaining term is nonnegative. This proves the cross-product inequality, including cases with zero total weight. Division by $ZZ^+ > 0$ gives the final assertion. □

For later use, interpret $w(s)$ as the number of choices of a specified group of positive parts having sum s, and $q(N-s)$ as the number of choices of all remaining parts. Replacing q by $q * B_b$ adds one more positive bounded part to the remaining group. Lemma 2.3 says that the sum of the specified group becomes stochastically smaller under the fixed-total condition. Lemma 2.1 verifies its hypothesis for every residual sequence used below.

3 Deletion and binomial normalization

Let $\mathcal{F} \subseteq 2^V$ be upward closed, where $|V| = n$, and let $d_k = \#\{D \in \mathcal{F} : |D| = k\}$. For $k \geq 1$ with $d_k > 0$, choose D uniformly from the k-element members of $\mathcal{F}$, then choose v uniformly from D. Write

$$R_k = \mathbb{P}(D \setminus \{v\} \in \mathcal{F}).$$

Lemma 3.1. *For $1 \leq k \leq n$ with $d_k > 0$,*

$$kd_kR_k = (n-k+1)d_{k-1}, \qquad R_k = \frac{d_{k-1}/\binom{n}{k-1}}{d_k/\binom{n}{k}}. \tag{8}$$

If $R_k \leq R_{k+1}$ whenever both are defined, then $(d_k)_{k=0}^n$ is ultra-log-concave.

Proof. Count pairs (A, D) with $A \in \mathcal{F}$, $|A| = k-1$, $|D| = k$, and $A \subset D$. Each A has exactly $n-k+1$ extensions, all in $\mathcal{F}$. Counting instead by (D, v) gives kd_kR_k, proving the first identity. The elementary ratio $\binom{n}{k}/\binom{n}{k-1} = (n-k+1)/k$ proves the second.

Fix $1 \le k < n$. If $d_{k-1} = 0$, the ultra-log-concavity inequality is automatic. Otherwise upward closure gives $d_k, d_{k+1} > 0$, and $R_k \le R_{k+1}$ is exactly $p_{k-1}/p_k \le p_k/p_{k+1}$, where $p_j = d_j/\binom{n}{j}$. Multiplying by the positive denominator proves the desired inequality. This also explains why the probabilistic proof never conditions on an empty family. □

For a nonempty graph, the empty set does not dominate. Hence $R_1 = 0$ whenever a dominating singleton exists. This will handle every transition involving a singleton without applying a two-gap deletion rule to it.

4 Cycle powers

Fix $n \ge 1$ and $\ell \ge 1$, and put

$$m = 2\ell + 1, \qquad P(z) = B_m(z).$$

A nonempty dominating set of C_n^ℓ, with a distinguished selected vertex, is encoded by its clockwise positive gaps $g_1, \dots, g_k$. They satisfy

$$1 \le g_i \le m, \qquad \sum_{i=1}^{k} g_i = n. \tag{9}$$

To justify the bound, consider the vertices strictly between consecutive selected vertices. They are covered by the two endpoints precisely when the clockwise distance is at most $2\ell+1$. If that distance is larger, the vertex $\ell+1$ steps after the left endpoint is too far from every selected vertex. The same argument applies to the final gap wrapping around the cycle. For a singleton, its one gap is n, and the condition $n \le 2\ell+1$ is exactly the condition that it dominates.

Let $T_k = [z^n]\, P(z)^k$. Each gap tuple satisfying (9) and each choice of its starting vertex determine one distinguished dominating set. Conversely, a dominating k-set has k choices of distinguished vertex. Therefore

$$kd_k(C_n^\ell) = nT_k. \tag{10}$$

This is a bijective pointing identity, not a count of rotation orbits; periodic gap tuples cause no exception. It also proves that a uniform dominating k-set followed by a uniform choice of a selected vertex gives a uniform tuple in (9).

For $k \ge 2$, deleting the distinguished selected vertex merges its two adjacent gaps. The remaining set dominates if and only if their sum is at most m. Thus, whenever $T_k > 0$,

$$R_k = \mathbb{P}(g_1 + g_2 \le m \mid 1 \le g_i \le m,\ \textstyle\sum_i g_i = n). \tag{11}$$

Exchangeability of the gaps permits the adjacent pair to be denoted by g_1, g_2. When $k = 2$, the merged gap has size n; the same condition correctly tests whether the remaining singleton dominates.

Apply Lemma 2.3 with coefficient sequences

$$w = [z^\bullet]P^2, \quad q = [z^\bullet]P^{k-2}, \quad p = [z^\bullet]P, \quad N = n, \quad L = m.$$

It gives $R_k \le R_{k+1}$ whenever the relevant counts are positive. The case $k = 2$ uses $q = \mathbf{1}_{\{0\}}$, already covered by Lemma 2.1. The case involving R_1 follows from $R_1 = 0$. Lemma 3.1 proves Theorem 1.1 for cycles.

These arguments also cover the small complete powers: if $n \le m$, the upper bound in (9) is automatic for every nonempty subset. The conventions for C_1 and C_2 give the same description. For $n = 0$ there is no index $1 \le k < n$ and the assertion is vacuous.

5 Path powers

Fix $n \ge 1$, $\ell \ge 1$, and put

$$a = \ell + 1, \qquad m = 2\ell + 1, \qquad A(z) = B_a(z), \qquad P(z) = B_m(z), \qquad N = n + 1.$$

In particular $a \le m$. Give the vertices of the path the labels $1, \dots, n$. For a nonempty selected set $1 \le v_1 < \cdots < v_k \le n$, define

$$g_0 = v_1, \qquad g_i = v_{i+1} - v_i \ (1 \le i < k), \qquad g_k = n + 1 - v_k.$$

The set dominates P_n^ℓ if and only if

$$1 \le g_0, g_k \le a, \quad 1 \le g_i \le m \ (1 \le i < k), \quad \sum_{i=0}^{k} g_i = N. \tag{12}$$

The endpoint conditions are exactly that vertices 1 and n are covered. Between two selected vertices, the same two-endpoint argument as for a cycle gives the interior bound. Conversely, these bounds cover every vertex, interval by interval. Taking cumulative sums reconstructs the selected set uniquely. In particular,

$$d_k(P_n^\ell) = [z^N]\, A(z)^2 P(z)^{k-1} \qquad (k \ge 1). \tag{13}$$

For $k = 1$ there are two endpoint gaps and no interior gaps; the formula retains both endpoint conditions and is still valid.

5.1 Two kinds of deletion

For $k \ge 2$ with $d_k > 0$, let E_k be the probability of $g_0 + g_1 \le a$ in the uniform gap model (12). This is the probability that the first selected vertex can be deleted. Reflection gives the same probability at the last selected vertex. For $k \ge 3$, let I_k be the probability of $g_1 + g_2 \le m$. All $k - 2$ interior selected vertices have this deletion probability, because the interior gaps are exchangeable. Hence

$$R_2 = E_2, \qquad R_k = \frac{2E_k + (k-2)I_k}{k} \quad (k \ge 3). \tag{14}$$

Lemma 5.1. *Whenever the required ranks are nonempty, $E_k \le E_{k+1}$ for $k \ge 2$ and $I_k \le I_{k+1}$ for $k \ge 3$.*

Proof. For the endpoint comparison use Lemma 2.3 with

$$w = [z^\bullet](AP), \quad q = [z^\bullet](AP^{k-2}), \quad p = [z^\bullet]P, \quad L = a.$$

The specified pair consists of one endpoint gap and one interior gap. The remaining group consists of the other endpoint and $k - 2$ interior gaps. For the interior comparison use

$$w = [z^\bullet]P^2, \quad q = [z^\bullet](A^2P^{k-3}), \quad p = [z^\bullet]P, \quad L = m.$$

Here the remaining group contains both endpoint gaps. The total is N in both applications. All residual sequences have property TP_2 by Lemma 2.1, including the smallest allowed values of k. □

5.2 Comparing an endpoint with the interior

Lemma 5.2. *For $k \geq 3$ with $d_k > 0$, one has $E_k \leq I_k$.*

Proof. For $b \geq 1$ define

$$t_b(z) = \sum_{s=1}^{b} (s-1) z^s.$$

There are $s-1$ ordered pairs of positive integers with sum s. Since $a \leq m$, restricting an endpoint–interior pair to sum at most a therefore gives the polynomial t_a. Restricting two interior gaps to sum at most m gives t_m. Thus

$$d_k E_k = [z^N]\, t_a A P^{k-2}, \tag{15}$$

$$d_k I_k = [z^N]\, t_m A^2 P^{k-3}. \tag{16}$$

The difference of their generating polynomials is $AP^{k-3}(At_m - Pt_a)$. Its second factor has nonnegative coefficients:

$$\begin{aligned} At_m - Pt_a &= \sum_{u=1}^{a} \sum_{v=1}^{m} (v-u) z^{u+v} \\ &= \sum_{u=1}^{a} \sum_{v=a+1}^{m} (v-u) z^{u+v}. \end{aligned} \tag{17}$$

The terms with $u, v \leq a$ cancel in pairs under $(u,v) \leftrightarrow (v,u)$, and every term remaining in (17) is nonnegative. The first factor AP^{k-3} also has nonnegative coefficients. Taking the coefficient of z^N and dividing by $d_k > 0$ proves the claim. □

5.3 Completion of the path proof

For the transition from two to three selected vertices, whenever those ranks exist, Lemmas 5.1 and 5.2 give

$$R_3 = \frac{2E_3 + I_3}{3} \geq E_3 \geq E_2 = R_2.$$

For $k \geq 3$ with both ranks nonempty, first increase the two deletion probabilities and then change their mixture weights:

$$\begin{aligned} R_{k+1} &\geq \frac{2E_k + (k-1)I_k}{k+1} \\ &= \frac{2E_k + (k-2)I_k}{k} + \frac{2(I_k - E_k)}{k(k+1)} \geq R_k. \end{aligned}$$

The transition from a singleton is covered by $R_1 = 0$. When $n = 1$ or $n = 2$, no nonexistent three-element rank is used: the required inequalities either are absent or involve $d_0 = 0$. Lemma 3.1 now proves Theorem 1.1 for paths.

No restriction on the size of ℓ relative to n has been imposed. When $\ell \geq n-1$, the graph is complete and the gap bounds are automatic for all nonempty subsets. The argument therefore includes complete path powers without changing (13). As before, the empty graph has no nontrivial inequality to verify.

6 Formalization

The companion project uses Lean 4.19.0 and a pinned mathlib revision. Its principal theorem is `DominationULC.galvin_zhang_conjecture`. The theorem `DominationULC.galvin_zhang_normalized` gives the two normalized inequalities. The declarations `DominationULC.path_domination_weighted` and `DominationULC.cycle_domination_weighted` give (1). The formal parameters also allow $\ell = 0$, when the graphs have no edges.

The formal graph definitions use the vertex set `Fin n`. Path adjacency requires distinct vertices u, v with $u \le v + \ell$ and $v \le u + \ell$. Cycle adjacency uses clockwise or anticlockwise distance at most ℓ. A dominating set contains a vertex equal or adjacent to each graph vertex. The count is the cardinality of a filtered finite set of vertex subsets.

The proof connects these graph definitions to the gap model. It uses the equivalence between compositions and their boundary sets. The path correspondence retains the two boundary gaps. For cycles, translation of subsets gives the pointing identity (10). Finite list enumeration then proves that the polynomial products count the required compositions.

The formal argument uses finite cross-products instead of probability spaces. In particular, (6) remains valid when a coefficient is zero. Euler's coefficient identity supplies the deletion algebra. For $r \ge 1$, write $\mathcal{E}Q = zQ'$ and $\mathcal{T}Q = \mathcal{E}Q - Q$. Then

$$(N - r)[z^N]\, P^r = r[z^N]\, (\mathcal{T}P)P^{r-1},$$

and $\mathcal{T}B_b = t_b$. The product rule gives the version with two endpoint factors. These identities avoid division by a potentially zero counting coefficient. The reproduction package gives the dependency versions, build commands, and axiom checks.

7 Relation to previous results

Theorem 1.1 proves the two-family statement in [1, Conjecture 1.8]. The same statement appears in [2, Conjecture 4.1.8, p. 57]. The proof combines bounded gaps with conditional deletion probabilities. The path case also needs the endpoint comparison (17). The gap encoding, convolution closure, and Toeplitz-matrix methods are established tools.

Consecutive-k-out-of-n:F reliability systems also use configurations with bounded runs. Drăgoi and Beiu study their Bernstein coefficients through alternating sums of positive terms [5]. They prove log-concavity properties of those positive terms. Such a property alone does not imply log-concavity of the resulting binomial-normalized sequence. Jianu, Dăuş, Drăgoi, and Beiu study zeros of circular reliability polynomials [6]. Their special real-rootedness results for consecutive-2 systems concern a different scope from the all-power inequalities proved here.

The theorem applies to every order and power of the two stated graph families. It makes no real-rootedness assertion and gives no corresponding conclusion for arbitrary graphs or all trees.

8 Research process and AI disclosure

This work is fully autonomous agentic research. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT. GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities; this check is not expert validation or peer review. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed.

Opus 5.5 by Anthropic used with Claude Code made the editorial revision of 1 November 2026 to the front matter, the abstract and this section; it did not change the mathematics. Section 6 states the formal scope and Section 7 the limits of the result.

References

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