The upper saturation spectrum of Berge triangles in triple systems

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Abstract

Let $E_n = \lfloor n^2/8 \rfloor$ be Győri's maximum number of edges in a Berge-triangle-free triple system on n vertices [6], and let $L_n = \lfloor (n-1)^2/8 \rfloor$. We show that for every $n \geq 8$ every Berge-triangle-free triple system with more than L_n edges is a spanning subsystem of a paired product $F(n,p)$: p disjoint pairs, $n-2p$ further vertices, and all triples formed by one pair and one further vertex. Consequently every saturated system with more than L_n edges is a paired product, and the part of the Berge-triangle saturation spectrum above L_n is exactly $\{p(n-2p) : 1 \leq p < n/2\} \cap (L_n, E_n]$. The threshold is sharp: for every $n \geq 8$ there is a saturated system with exactly L_n edges that is not a paired product. The upper window has $\lfloor n/4 \rfloor$ or $\lfloor n/4 \rfloor + 1$ integers and contains $n/4 - O(\sqrt{n})$ sizes that are not in the spectrum, given explicitly by quadratic deficits. Bushaw, English, Heath, Johnston and Rombach [1] proposed the study of this spectrum; the result determines it above L_n and shows that the number of sizes missing near the maximum is unbounded. The proof uses the own-pair reduction of Frankl, Füredi, Goorevitch, Holzman and Simonyi [4], a rigidity lemma for dense bipartite own-pair graphs, and the equality case of Erdős's bound for nonbipartite triangle-free graphs [3]. A Lean 4 project proves that every paired product is Berge-triangle-free and saturated with $p(n-2p)$ edges, proves the sharpness construction for every $n \geq 8$ and the arithmetic of the window, and derives the other statements from the containment theorem, whose proof is conventional; exact checkers verify the constructions and the arithmetic for small orders.

1 Definitions and results

A *triple system* on a vertex set V is a set H of 3-element subsets of V, its *edges*. All vertices of V count, including isolated ones. H contains a *Berge triangle* if there are distinct vertices x, y, z and distinct edges $h_1, h_2, h_3 \in H$ with $\{x, y\} \subseteq h_1$, $\{y, z\} \subseteq h_2$ and $\{z, x\} \subseteq h_3$; copies need not be induced. H is *saturated* if it contains no Berge triangle but $H \cup \{T\}$ contains one for every 3-set $T \notin H$. The *saturation spectrum* $\mathrm{ES}_3(n, \text{Berge-}K_3)$ is the set of sizes $|H|$ of saturated systems on n vertices.

Győri [6] proved that a Berge-triangle-free triple system on n vertices has at most $E_n = \lfloor n^2/8 \rfloor$ edges, and Frankl, Füredi, Goorevitch, Holzman and Simonyi [4] characterized the extremal systems. Saturation numbers of Berge triangles were studied in [2]. Bushaw, English, Heath, Johnston and Rombach [1, Section 6], after determining spectra of Berge stars, proposed the spectrum of the Berge triangle as a further problem, recording its minimum $\lceil (n-1)/2 \rceil$ and maximum E_n, and suspected that only finitely many values near the top are missing. Their remark is a suspicion rather than a numbered conjecture.

For $1 \leq p < n/2$ the *paired product* $F(n,p)$ has vertex set $P_1 \cup \cdots \cup P_p \cup B$, where the P_i are disjoint pairs and $|B| = n - 2p \geq 1$, and edge set $\{P_i \cup \{b\} : 1 \leq i \leq p,\ b \in B\}$. It has $p(n-2p)$ edges; for suitable p this attains E_n. Write

$$E_n = \left\lfloor \frac{n^2}{8} \right\rfloor, \qquad L_n = \left\lfloor \frac{(n-1)^2}{8} \right\rfloor, \qquad W_n = E_n - L_n.$$

Theorem 1 (dense containment)**.** *Let $n \geq 8$. Every Berge-triangle-free triple system H on n vertices with $|H| > L_n$ is a spanning subsystem of a paired product $F(n,p)$.*

Theorem 2 (upper spectrum)**.** *Let $n \geq 8$. A saturated triple system on n vertices with more than L_n edges is isomorphic to a paired product. Hence*

$$\mathrm{ES}_3(n, \textit{Berge-}K_3) \cap (L_n, E_n] = \{p(n-2p) : 1 \leq p < n/2\} \cap (L_n, E_n].$$

Theorem 3 (sharpness)**.** *For every $n \geq 8$ the largest saturated triple system on n vertices that is not isomorphic to a paired product has exactly L_n edges. In particular the strict threshold in Theorems 1 and 2 cannot be lowered.*

Corollary 4 (explicit deficits)**.** *Let $n \geq 8$ and $0 \leq d < W_n$. Then $E_n - d$ belongs to the saturation spectrum if and only if*

- *$d = 2j^2$ for some integer $j \geq 0$, when $n \equiv 0 \pmod 4$;*
- *$d = 2j(j+1)$ for some integer $j \geq 0$, when $n \equiv 2 \pmod 4$;*
- *$d = j(j+1)/2$ for some integer $j \geq 0$, when n is odd.*

Here $W_n = \lfloor n/4 \rfloor$, except that $W_n = \lfloor n/4 \rfloor + 1$ when $n \equiv 3 \pmod 4$. The window $(L_n, E_n]$ therefore contains $n/4 - O(\sqrt{n})$ sizes outside the spectrum.

For example, above $L_{16} = 28$ only 30 and 32 occur, and for $n = 400$ only 8 of the 100 sizes $19901, \ldots, 20000$ occur (Table 1). The number of missing sizes is thus unbounded, and they are not confined to a bounded distance from the maximum. This rules out the natural uniform-in-n reading of the suspicion in [1]. The spectrum at or below L_n is not determined here.

Prior work. The paired products, their size, the bound E_n and the classification of the extremal systems are known [6, 4]. Lemma 5 below is the own-pair reduction of [4, Observation 2]. The nonbipartite bound in Lemma 7 is due to Erdős [3, Lemma 1]; we include its proof because we need its equality case. Gerbner [5] obtained qualitative matching and shadow structure for near-extremal Berge-triangle-free systems; such structure may already imply, after a saturation argument, that the spectrum misses many sizes near the top. What we add is the exact finite threshold L_n for every $n \geq 8$, the nonproduct construction at the threshold, and the explicit spectrum in the whole window. We did not find these statements in the searched literature. The full text of a 2026 preprint of Xu, Zeng and Zhang on the Turán number of a matching together with a Berge triangle (arXiv:2607.12300) could not be inspected, and its structural lemmas may bear on Theorem 1. No priority claim is made.

2 Own pairs

A pair of vertices is an *own pair* of an edge h if h is the only edge containing it. The *shadow* ∂H is the graph of all pairs contained in some edge.

Lemma 5 ([4, Observation 2])**.** *Let H be Berge-triangle-free. Every edge has at least two own pairs. Choosing exactly two own pairs from each edge gives a triangle-free graph G with $e(G) = 2|H|$. Every triangle of ∂H is an edge of H.*

Proof. If $h = \{a, b, c\}$ had non-own pairs ab and ac, pick edges $h_1 \neq h \ni a, b$ and $h_2 \neq h \ni a, c$. They are distinct, since an edge containing ab and ac equals h, and h_1, h_2, h with the pairs ab, ac, bc form a Berge triangle. Own pairs chosen from different edges are distinct, so $e(G) = 2|H|$. If abc is a triangle of ∂H but not an edge, the edges containing ab, bc, ca are pairwise distinct (a common edge of two of them would be abc) and form a Berge triangle. Finally, a triangle of G is a triangle of ∂H, hence an edge h; but only two of its pairs were chosen from h, and the third, being own, cannot be chosen from any other edge. □

Fix such a *selected own-pair graph* G, and let M be the set of *omitted pairs*, the third pair of each edge. A selected pair is own and is selected only from its own edge, where it is not omitted, so $M \cap G = \emptyset$ and $\partial H = G \cup M$. Omitted pairs may have codegree one.

3 Dense bipartite own-pair graphs

Lemma 6 (bipartite rigidity)**.** *Let $H \neq \emptyset$ be Berge-triangle-free with a selected own-pair graph G that is bipartite with parts A, B, $a = |A|$, $b = |B|$, and suppose $D = ab - e(G) < \min(a, b)$. Then H is a spanning subsystem of a paired product.*

Proof. The two selected pairs of an edge form a path of length two in G, so each omitted pair lies inside A or inside B.

M is a matching. Suppose $xy, xz \in M$ lie in A. If some $w \in B$ were G-adjacent to x, y, z, then xyw and xzw would be triangles of ∂H, hence edges by Lemma 5, and the selected pair xw would lie in two edges. So each $w \in B$ misses one of xw, yw, zw, and $D \geq b$, a contradiction. The same holds inside B.

Let p and q be the numbers of pairs of M inside A and B, and $r = a - 2p$, $s = b - 2q$ the numbers of unmatched vertices.

Blocks. Let $P \subseteq A$ and $Q \subseteq B$ be pairs of M. An edge supplying a G-edge between P and Q has its omitted pair in M meeting P or Q, so it is $P \cup \{v\}$ with $v \in Q$ or $Q \cup \{u\}$ with $u \in P$, and contributes two G-edges to this 2×2 block. Two such edges of opposite kinds share a cross pair, which is selected and own; two of the same kind, say $P \cup \{u\}$ and $P \cup \{v\}$ with $Q = \{u, v\}$, together with an edge $h \supseteq Q$ (which exists since $Q \in M$) and any $x \in P$, give the Berge triangle on x, u, v. Hence each block has at most two G-edges. An unmatched $u \in A$ and an unmatched $v \in B$ are not G-adjacent: the edge supplying uv would have an omitted pair at u or v. Counting these disjoint sets of missing cross pairs,

$$D \geq 2pq + rs.$$

If $p, q \geq 1$ and $r \leq s$ then $rs \geq r$ and $2pq \geq 2p$, so $D \geq a$; symmetrically $D \geq b$ if $s \leq r$. So p or q is zero, and since $M \neq \emptyset$ we may assume $p > 0 = q$. Then every omitted pair lies in A, and an unmatched vertex of A would have G-degree zero, forcing $D \geq b$. So $r = 0$: A is partitioned by the pairs of M, and every edge is a pair of M together with a vertex of B. As $b > D \geq 0$, this is a paired product containing H. □

4 The nonbipartite threshold and its equality case

Let $T_n = \lfloor (n-1)^2/4 \rfloor + 1$.

Lemma 7 (Erdős [3, Lemma 1])**.** *A nonbipartite triangle-free graph on n vertices has at most T_n edges.*

Proof. Let C be a shortest odd cycle, of length $2k + 1 \geq 5$; it has no chords. A vertex outside C has at most two neighbours on C: three neighbours split C into gaps of length at least two, one gap is odd and at most $2k - 3$, and it closes a shorter odd cycle through the vertex. With $m = n - 2k - 1$ and Mantel's theorem outside C,

$$e(G) \leq (2k+1) + 2m + \left\lfloor \frac{m^2}{4} \right\rfloor = T_n - (k-2)(n-k-1) \leq T_n,$$

where the identity uses $\lfloor (m+2k)^2/4 \rfloor = \lfloor m^2/4 \rfloor + km + k^2$. □

Lemma 8. *Let $n = 4t + 3 \geq 11$, and let G be nonbipartite and triangle-free with $e(G) = T_n$. Then $V(G)$ is partitioned into $A, D, C, \{u\}, \{v\}$ with $|C| = 2t$, $|A| + |D| = 2t + 1$, $|A|, |D| \geq 1$, and the edges of G are exactly those between C and $A \cup D$, those between u and A, those between v and D, and uv.*

Proof. Equality forces $k = 2$, two neighbours on the 5-cycle $c_0 \dots c_4$ for every outside vertex, and equality in Mantel's theorem outside the cycle; so the $4t - 2$ outside vertices span a complete bipartite graph with parts U, V of size $2t - 1$. An outside vertex adjacent to c_{i-1}, c_{i+1} has *type* i. For adjacent $x \in U$, $y \in V$ the neighbourhoods on the cycle are disjoint, which happens exactly when their types are adjacent indices of the cycle. As the 5-cycle contains no $K_{2,2}$, one side, say U, has a single type, which we rotate to 0. Then V splits into V_1, V_4 by type, and $C = \{c_0\} \cup U$, $A = \{c_1\} \cup V_1$, $D = \{c_4\} \cup V_4$, $u = c_2$, $v = c_3$ have the stated properties. □

Lemma 9 (switching)**.** *If a Berge-triangle-free H has a selected own-pair graph of the form in Lemma 8, then another selection of own pairs gives a bipartite graph with the same number of edges.*

Proof. Let h be the edge supplying uv. Its other selected pair contains u or v; by symmetry $h = \{u, v, x\}$ with ux selected, so $x \in A$ and vx is omitted. The pair vx is also own: another edge containing vx would have vx as its omitted pair and a common G-neighbour of v and x as third vertex, and u is their only common neighbour. Replacing uv by vx keeps two own pairs per edge and gives a graph bipartite with parts $A \cup D$ and $C \cup \{u, v\}$. □

5 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $|H| > L_n$ and select an own-pair graph G; it is triangle-free with $e(G) = 2|H|$. Comparing residues of $(n-1)^2$ modulo 8 gives $2|H| \geq 2L_n + 2 \geq T_n$, with equality only if $n \equiv 3 \pmod 4$ and $|H| = L_n + 1$. If $e(G) > T_n$, then G is bipartite by Lemma 7; if $e(G) = T_n$ and G is not bipartite, Lemmas 8 and 9 give a bipartite selection with the same number of edges. So some selected own-pair graph G is bipartite with $e(G) \geq T_n$. If its parts have sizes $a \leq b$, then $a(b-1) \leq \lfloor (n-1)^2/4 \rfloor$, so $D = ab - e(G) \leq ab - T_n \leq a - 1 < \min(a, b)$, and Lemma 6 applies. □

Lemma 10. *Every paired product $F(n, p)$ is Berge-triangle-free and saturated.*

Proof. Each edge has exactly one vertex in B. A Berge triangle cannot contain two vertices of B, since one edge would contain both. If it contains none, the pair xy forces x, y into one pair P_i, and likewise for yz, so three distinct vertices lie in P_i. If it contains one, say $z \in B$, then $x, y \in P_i$ and both edges through z equal $P_i \cup \{z\}$. For saturation, an absent triple contains two vertices of B, which are joined by the two-edge Berge path b, x, b' through any $x \in P_1$, or vertices from two different pairs, joined through any $b \in B$; the only other triples, one pair and one vertex of B, are present. Adding the triple closes the path into a Berge triangle. □

Proof of Theorem 2. If H is saturated with $|H| > L_n$, then $H \subseteq F$ for a paired product F by Theorem 1. Adding any triple of $F \setminus H$ to H stays inside the Berge-triangle-free F, so saturation forces $H = F$. Conversely every $F(n, p)$ is saturated by Lemma 10. □

6 The sharpness construction

For $p \geq 1$ and $q \geq 3$ take disjoint pairs $P_1 = \{x, y\}, P_2, \dots, P_p$, a set S of $q - 2$ vertices and two further vertices u, v, so $n = 2p + q$. Let $J(p, q)$ consist of

$$P_i \cup \{z\} \ (1 \leq i \leq p,\ z \in S), \qquad P_i \cup \{v\} \ (2 \leq i \leq p), \qquad \{x, u, v\}.$$

It has $p(q-2) + (p-1) + 1 = p(q-1)$ edges.

Lemma 11. *$J(p,q)$ is saturated. For $p \geq 2$ it is not isomorphic to a paired product.*

Proof. Freeness. Without $\{x,u,v\}$ the system is contained in the paired product with pairs P_i and base $S \cup \{u,v\}$, so a Berge triangle must use $\{x,u,v\}$ for a pair inside it, and the other two edges form a two-edge Berge path between the two vertices of that pair. Since u lies in no other edge, the pair is xv. A first edge at x is $P_1 \cup \{z\}$ with $z \in S$, and neither y nor z lies in an edge with v.

Saturation. The vertex u has a two-edge Berge path to every vertex outside $\{x,u,v\}$: to y or $z \in S$ through x, using $\{x,u,v\}$ and $P_1 \cup \{z\}$, and to a vertex of P_i, $i \geq 2$, through v, using $\{x,u,v\}$ and $P_i \cup \{v\}$. An absent triple containing u contains such a vertex. For an absent triple avoiding u: two vertices in different pairs are joined through any $z \in S$; two vertices of S are joined through x, and so are v and $z \in S$, using $\{x,u,v\}$ and $P_1 \cup \{z\}$. Otherwise the triple is a pair P_i with one vertex of $S \cup \{v\}$; all of these are present except $P_1 \cup \{v\}$, whose pair yv closes the path y,x,v.

Not a product. The vertex u has degree one, and for $p \geq 2$ the edges $P_1 \cup \{z\}$ and $P_2 \cup \{v\}$ are disjoint. In a paired product a vertex of degree one forces $p = 1$ or $|B| = 1$, and then all edges share a vertex. □

Proof of Theorem 3. Take $p = \lfloor n/4 \rfloor$ and $q = n - 2p$. For $n \geq 8$ we have $p \geq 2$ and $q \geq 3$, and checking the four residues of n modulo 4 gives $p(q-1) = p(n-2p-1) = L_n$. So $J(p,q)$ is a saturated system with L_n edges that is not a paired product. By Theorem 2 no larger nonproduct saturated system exists. Since $J(p,q)$ is saturated, it is not contained in any paired product (saturation would force equality), so the threshold of Theorem 1 is sharp as well. □

The construction proves the value L_n outside the product class; it does not classify the nonproduct saturated systems with L_n edges, and other such systems exist, for instance at $n = 11$.

7 Arithmetic of the window

Write $n = 4t + r$. Parametrizing p around $n/4$ gives

$$\begin{aligned}
n = 4t,\ p = t + j: &\quad E_n - p(n-2p) = 2j^2, \\
n = 4t+2,\ p = t + j: &\quad E_n - p(n-2p) = 2j(j-1), \\
n = 4t+1,\ p = t + j: &\quad E_n - p(n-2p) = 2j^2 - j, \\
n = 4t+3,\ p = t + 1 + j: &\quad E_n - p(n-2p) = 2j^2 + j,
\end{aligned}$$

with $j \in \mathbb{Z}$. As j ranges over the integers, the nonnegative values of $2j(j-1)$ are the numbers $2k(k+1)$, and those of $2j^2 \mp j$ are the triangular numbers $k(k+1)/2$, $k \geq 0$. If $d < W_n$ then $E_n - d > L_n > 0$, so a solution of $p(n-2p) = E_n - d$ automatically has $0 < p < n/2$. With Theorem 2 this proves Corollary 4. The values E_n, L_n are $2t^2$, $2t^2 - t$ for $n = 4t$; $2t^2 + t$, $2t^2$ for $n = 4t+1$; $2t^2 + 2t$, $2t^2 + t$ for $n = 4t+2$; and $2t^2 + 3t + 1$, $2t^2 + 2t$ for $n = 4t+3$, which gives W_n. Each quadratic family meets the window in $O(\sqrt{W_n})$ values.

8 Verification

Exact checks. The program `check_spectrum.py` in `certificate/`, which uses only the Python standard library, builds $J(\lfloor n/4 \rfloor, n - 2\lfloor n/4 \rfloor)$ and every paired product above the threshold for $8 \leq n \leq 20$, and checks freeness, saturation, sizes and product recognition by direct search. It compares the product sizes in $(L_n, E_n]$ with Corollary 4 and the formula for W_n for $8 \leq n \leq 2000$, and recomputes Table 1. It also checks four witness systems directly: $J(2,7)$ at $n = 11$, $J(4,8)$ at $n = 16$, and paired products with 30 and 32 edges at $n = 16$. A separate

n	L_n	E_n	sizes in $(L_n, E_n]$ in the spectrum	missing sizes
8	6	8	8	7
11	12	15	14, 15	13
16	28	32	30, 32	29, 31
40	190	200	192, 198, 200	191, 193–197, 199
400	19900	20000	19902, 19928, 19950, 19968, 19982, 19992, 19998, 20000	92 of the 100 sizes

Table 1: The upper window for some orders.

checker validates the stored search-free certificates of these witnesses, listing for every absent triple a closing Berge path.

Lean. The Lean 4.35.0 project in `lean/`, pinned to Mathlib commit `56a0058`, defines triple systems on an arbitrary vertex type, Berge triangles, freeness, saturation and paired products as in Section 1. It proves Lemma 10 on every vertex type, with $p(n-2p)$ edges on n vertices. It defines $J(p, n-2p)$ on the vertex set $\{0, \dots, n-1\}$, with $P_i = \{2i-2, 2i-1\}$, $u = 2p$, $v = 2p+1$ and $S = \{2p+2, \dots, n-1\}$, and proves Lemma 11 together with the size $p(n-2p-1)$. With $p = \lfloor n/4 \rfloor$ this gives, for every $n \geq 8$, the lower bound of Theorem 3 and a free system with L_n edges in no paired product. For every n the project proves the formula for W_n, the description in Corollary 4 of the product sizes $E_n - d$ with $d < W_n$, and that at most $\lfloor\sqrt{2n}\rfloor/2 + 1$ sizes in $(L_n, E_n]$ have the form $p(n-2p)$, so that at least $\lfloor n/4 \rfloor - \lfloor\sqrt{2n}\rfloor/2 - 1$ do not. Theorem 1 is not proved in Lean. Its statement at order n enters as a hypothesis, from which the project derives Theorem 2, the upper bound of Theorem 3, Corollary 4 for the spectrum and Győri's bound E_n. The 33 audited declarations depend only on `propext`, `Classical.choice` and `Quot.sound`.

9 Limitations and open questions

The spectrum at or below L_n is open, and the nonproduct saturated systems with exactly L_n edges are not classified. For $n \leq 7$ the statements are not claimed. The literature comparison is bounded, as described in Section 1.

10 Research process and AI disclosure

This is fully autonomous agentic research. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed, no expert validation is claimed, and the novelty assessment is provisional.

The problem selection, the theorems, their proofs, the construction, the exact checkers, the witness certificates and the literature search were produced by autonomous AI research agents in an earlier session, which also ran internal AI-assisted reviews of the proof; these reviews are not peer review. Its records do not name the model used for the research; they describe the internal reviews as Astra reviews. During that session the target threshold was strengthened from $E_n - \lfloor n/4 \rfloor$ to L_n, which differs from it by one edge when $n \equiv 3 \pmod 4$, and the containment statement was extended from saturated to all free systems.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It checked each proof step, wrote its own exact checker for the constructions and the arithmetic, reran the certificate checker, wrote the Lean project and ran its fresh build and axiom audit, and wrote this manuscript. It also made the editorial revision of 1 November 2026, which did not change the mathematics. AI checks are not human peer review. Section 8 states the formal scope and Section 9 the remaining limits.

References

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