The small-coefficient upper half of the Keller–Klein conjecture

Analytic completion and Lean 4 formalization

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Abstract

Let $X = \sum_{i=1}^{n} v_i \varepsilon_i$, where the signs are independent and uniform, $\sum_i v_i^2 = 1$, and $|v_i| \le a \le 7/100$. For every $x \ge 0$ we prove

$$\mathbb{P}(|X| \le x) \le 2\Phi(x) + 2\Phi(a) - 2, \qquad \mathbb{P}(X \le x) - \Phi(x) \le \Phi(a) - \tfrac{1}{2}.$$

Here $a > 0$ and Φ is the standard normal distribution function. This is the upper half of Conjecture 9.3 of Keller and Klein [1] in the small-coefficient regime that the recent manuscript of Ivanisvili [2] explicitly leaves open. The proof handles arbitrary unequal real coefficients and atoms at both endpoints. It is analytic, using Prawitz smoothing, coefficient profiles and a compensated alias estimate, and it carries out the route proposed in Section 11 of [2]: a small-a analytic upper bound mirroring its Section 8. Prawitz's method, Ivanisvili's envelopes, Hölder profile reduction and oscillation inequality, and the previously treated coefficient ranges are credited to their sources. Together with Theorems 1.1 and 10.9 of [2], the theorem implies the full Keller–Klein conjecture; those external results are computer-assisted, have not been refereed and are not part of the Lean development, so the full conjecture follows here only conditionally on them. A Lean 4 formalization proves the theorem itself for the actual finite sign distribution, including the characteristic function, the smoothing inequality, the analytic estimates and the identification with Mathlib's normal distribution function. Public literature searches support the novelty of this completion, but do not establish priority against unpublished or simultaneous work.

1 Introduction and main theorem

In their proof of Tomaszewski's conjecture, Keller and Klein conjectured that a normalized Rademacher sum whose largest coefficient magnitude is a lies within Kolmogorov distance $\Phi(a) - \frac{1}{2}$ of the standard normal law [1, Conjecture 9.3]. Ivanisvili's computer-assisted manuscript of 21 October 2026 [2] asserts the lower half of this inequality for every coefficient vector and the full inequality for $a \ge 0.07$, and leaves the upper half for $a < 0.07$ open. Theorem 1.1 below supplies that upper half. Section 2 gives the precise relation to the conjecture and to prior work.

Throughout we write

$$\varphi_0 = (2\pi)^{-1/2}, \quad g(t) = e^{-t^2/2}, \quad \varphi(t) = \varphi_0 g(t), \quad c(t) = \int_0^t \varphi(s)\, ds = \Phi(t) - \tfrac{1}{2}, \quad H(t) = 2c(t).$$

For $t \ge 0$, $H(t) = \mathbb{P}(|Z| \le t)$ for $Z \sim N(0,1)$.

Theorem 1.1 (Small-coefficient upper concentration)**.** *For every integer $n \geq 0$, real vector $(v_i)_{i=1}^n$, and real numbers a, x satisfying*

$$0 < a \leq \frac{7}{100}, \qquad |v_i| \leq a \ (1 \leq i \leq n), \qquad \sum_{i=1}^{n} v_i^2 = 1, \qquad x \geq 0,$$

the uniform independent-sign sum $X = \sum_i v_i \varepsilon_i$ satisfies

$$\mathbb{P}(|X| \leq x) \leq H(x) + H(a). \tag{1}$$

Consequently,

$$\mathbb{P}(X \leq x) - \Phi(x) \leq c(a) \qquad (x \geq 0). \tag{2}$$

The parameter a may be any upper bound on the maximum coefficient magnitude; it need not equal an attained coefficient. In particular it may be chosen as $\max_i |v_i|$. No ordering, rationality, distinct-sums assumption, positive lower coefficient bound, or restriction on n is imposed. Normalization rules out $n = 0$. Zero coefficients are allowed. For a concrete nonvacuity witness, $n = 256$, $v_i = 1/16$, and $a = 1/16$ satisfy all coefficient hypotheses. This witness is formally checked without enumerating 2^{256} outcomes.

The closed interval in (1) is essential. Symmetry gives the exact identity, including atoms,

$$\mathbb{P}(|X| \leq x) = 2\mathbb{P}(X \leq x) - 1 \qquad (x \geq 0).$$

Thus (1) and (2) are equivalent. The proof below does not replace either event by a strict event without accounting for its endpoint.

What is new and what is checked. The proposed original contribution is the analytic completion of the upper half for every coefficient vector with $a \leq 7/100$. Prawitz smoothing [6] is classical, and the symmetric smoothing bound, the characteristic-function envelopes, the Hölder profile covering and the oscillation inequality are taken from or modelled on Ivanisvili's manuscript [2]; each is credited where it is used. The Lean development described in Section 6 proves Theorem 1.1 for the actual finite law of X, with no analytic step left as a hypothesis. The consequence for the full conjecture, Corollary 2.1, is a conventional implication from Ivanisvili's computer-assisted theorems, which are not formalized.

2 Relation to the established open problem

Keller and Klein's Conjecture 9.3 asks, for a normalized Rademacher sum and $a = \max_i |v_i|$, whether

$$\sup_{t \in \mathbb{R}} \left|\mathbb{P}(X \leq t) - \Phi(t)\right| \leq c(a). \tag{3}$$

It appears in the paper proving Tomaszewski's conjecture [1]. These are different assertions: Tomaszewski's conjecture is already a theorem and is not the novelty target here.

For clarity, distinguish the following previously studied regimes.

- Equal coefficients were already covered by normal-approximation results for symmetric binomial and hypergeometric laws. Keller and Klein explicitly cite Mattner–Schulz for this case [4]; see also Hipp–Mattner [5].

- Ma, Ye, and Han prove the conjecture for $n \le 5$, treat equal weights, and give a large-coefficient upper-half result with additional hypotheses [3]. Their theorem does not cover the unequal small coefficients in Theorem 1.1.
- Ivanisvili's manuscript dated 21 October 2026 asserts the lower discrepancy for every coefficient vector and the full conjecture for $a \ge 0.07$ [2]. Its public index explicitly leaves $a < 0.07$ open. The manuscript states that its proof was generated by Grok Bot using Cursor's agents; it is a computer-assisted manuscript, not a refereed journal publication. Its Section 11 states that reaching $a \to 0$ in the upper half needs a new idea, and proposes a small-a analytic upper bound mirroring its Section 8. The present proof carries out that proposal.

The new theorem supplies exactly its missing small-a upper half. Its significance is the closure of that remaining regime, including all unequal coefficient vectors and all thresholds, with the conjectured nonlinear error $c(a)$. This is not a finite-dimensional enumeration or a change in the conjecture's assumptions.

Corollary 2.1 (The full conjecture, conditional on external results)**.** *If Ivanisvili's Theorems 1.1 and 10.9 hold, Theorem 1.1 implies (3) for every normalized finite Rademacher sum.*

Proof. For $a \ge 7/100$, use the cited full-range theorem. For $a \le 7/100$, the new upper discrepancy and the cited lower discrepancy bound both sides for $t \ge 0$. For $t < 0$, put $y = -t > 0$. Symmetry gives $\mathbb{P}(X \le t) = 1 - \mathbb{P}(X < y)$. Taking limits from below in the two bounds at nonnegative locations $r \uparrow y$, and using continuity of Φ, gives the bounds for $\mathbb{P}(X < y)$ and hence for t. At $t = 0$, the upper bound is already included in the new theorem and the lower bound follows from symmetry. □

Verification boundary. The corollary is an ordinary implication using external results. The Lean project proves Theorem 1.1, not those external results or the all-coefficient two-sided statement (3). Neither a cited theorem nor a numerical certificate is inserted as a new axiom in the formal proof.

The sharpness claim also has a precise scope. The familiar equal-weight example in [1] has odd n, $v_i = 1/\sqrt{n}$, and $m = 1/\sqrt{n}$. Every negative value of X is at most $-m$, so

$$\mathbb{P}(X \le -m) - \Phi(-m) = \tfrac{1}{2} - \Phi(-m) = c(m).$$

Thus the full two-sided radius is attained on an infinite sequence of maximum coefficients tending to zero. This does not assert pointwise optimality for every real coefficient cap, or equality in the new upper half restricted to nonnegative thresholds. The equal-weight sharpness example is prior mathematics, not part of the originality claim.

3 Exact smoothing and the characteristic function

Changing each negative coefficient's sign is a bijection of the Boolean cube, so assume $v_i \ge 0$ for the analytic argument. Put

$$f(s) = \mathbb{E}\cos(sX) = \prod_i \cos(v_i s), \qquad \rho(u) = (1-u)\cot(\pi u) + \frac{1}{\pi} \quad (0 < u < 1),$$

and, for $\xi \ge 0$, define

$$k_\xi(u) = (1-u)\cos(\pi\xi u) + \rho(u)\sin(\pi\xi u).$$

The kernel's endpoint values may be filled by its finite limits; changing them does not change any Lebesgue integral. The Lean definitions use raw division at endpoints, and the integral arguments explicitly use almost-everywhere equality where needed.

The symmetric Prawitz bound at $T = \pi/a$ is

$$\mathbb{P}(|X| \le x) \le 2 \int_0^1 k_{x/a}(u) f(\pi u/a)\, du. \tag{4}$$

Prawitz smoothing is prior mathematics [6], and this symmetric form is Ivanisvili's Corollary 3.2 and Lemma 10.1 [2]; the project proves the particular majorant used here, rather than postulating it.

For completeness, its analytic construction can be checked directly. Set

$$A(z) = 2 \int_0^1 \rho(u) \sin(2\pi z u)\, du, \qquad S(z) = \frac{\sin^2(\pi z)}{\pi^2}.$$

We use the unnormalized convention $\operatorname{sinc}(t) = \sin(t)/t$ for $t \neq 0$, with $\operatorname{sinc}(0) = 1$. Trigonometric addition and an elementary triangular cosine integral give, for $z > 0$,

$$A(z+1) - A(z) = \frac{S(z)}{z^2(z+1)^2}, \qquad A(m) = 1 \quad (m \in \mathbb{N},\ m \ge 1).$$

The sine-difference identity gives

$$A(z+m) - A(m) = 4 \int_0^1 \rho(u) \sin(\pi z u) \cos\big((2\pi m + \pi z)u\big)\, du.$$

The factor $\rho(u)\sin(\pi z u)$ is integrable: the sine removes the $1/(\pi u)$ singularity at zero, and the limit at the other endpoint is finite. The Riemann–Lebesgue lemma therefore gives $A(z+m) \to 1$ as $m \to \infty$. The first displayed recurrence makes $A(z+m)$ nondecreasing. It makes $A(z+m) + S(z)/(z+m)^2$ nonincreasing, since its increment is $-2S(z)/((z+m)(z+m+1)^2)$. Both sequences tend to 1, so

$$A(z) \le 1 \le A(z) + \operatorname{sinc}^2(\pi z), \qquad z > 0.$$

Oddness of A and its value at zero now yield

$$\mathbf{1}_{[0,\infty)}(w) \le \frac{1 + A(w/(2\pi)) + \operatorname{sinc}^2(w/2)}{2}, \qquad w \in \mathbb{R}.$$

The right side equals 1 at $w = 0$. Apply the majorant at $T(x - X)$ and $T(x + X)$, subtract 1, and average. The identity $\operatorname{sinc}^2(w/2) = 2\int_0^1 (1-u)\cos(uw)\, du$ and the exact cosine product for the uniform cube prove (4).

Two bounds on this integral are used. First, when $0 \le \xi \le 1/2$, the kernel is nonnegative and weighted arithmetic–geometric mean gives

$$|f(s)| \le \sum_{i : v_i > 0} v_i^2 |\cos(v_i s)|^{1/v_i^2}. \tag{5}$$

The weights sum to 1. Therefore it suffices in this range to bound uniformly the actual profile integrals

$$G(a, b, \xi) = \int_0^1 k_\xi(u) |\cos(\pi b u/a)|^{1/b^2}\, du, \qquad 0 < b \le a.$$

This implements the Hölder covering of Ivanisvili's Lemma 10.3 [2] in an elementary finite form; he notes that the covering is sharp at $x = 0$, where it recovers his Proposition 9.1.

For the remaining thresholds use the characteristic-function envelopes. There is a root $\theta \in (17/10, 9/5)$ of $|\cos\theta| = g(\theta)$ with $\pi/2 < \theta < \pi$. Define

$$L_a(s) = \begin{cases} \cos(as)^{1/a^2}, & as \le \pi/2, \\ -|\cos(as)|e^{-(1-a^2)s^2/2}, & \pi/2 < as \le \theta, \\ -|\cos(as)|^{1/a^2}, & \theta < as \le \pi, \end{cases} \quad U_a(s) = \begin{cases} g(s), & as \le \theta, \\ |\cos(as)|^{1/a^2}, & \theta < as \le \pi. \end{cases}$$

Then $L_a \le f \le U_a$ on $[0, \pi/a]$. One verifies this using the monotonicity of $\log|\cos y|/y^2$: it decreases before $\pi/2$ and increases after it. Up to θ, each absolute cosine is bounded by a Gaussian. For a negative product in the middle interval, remove a negative factor, bound its magnitude by $|\cos(as)|$, and bound the remaining factors by the Gaussian with variance at least $1 - a^2$. These arguments prove the stated envelopes for every coefficient vector; the root and its bounds are also proved in Lean.

Define

$$J_a(x) = \int_0^1 \max\{k_{x/a}(u)L_a(\pi u/a), k_{x/a}(u)U_a(\pi u/a)\}\, du. \tag{6}$$

The maximum handles either sign of the kernel. Thus $\mathbb{P}(|X| \le x) \le 2J_a(x)$ without a positivity assumption on that kernel.

4 Small thresholds: the uniform profile bound

We prove

$$G(a, b, \xi) \le c(a) + c(\xi a), \quad 0 < a \le 7/100, \quad 0 < b \le a, \quad 0 \le \xi \le 1/2. \tag{7}$$

Put $r = b/a$. Throughout, use the proved elementary constants

$$3 < \pi, \quad 9 < \pi^2 < 10, \qquad 39/100 < \varphi_0 < 2/5.$$

4.1 Kernel and primary peak

For $y = \pi u \in (0, \pi)$,

$$\frac{1}{y} - \frac{y}{2} \le \rho(u) \le \frac{1}{y} - \frac{y}{3} + \frac{y^2}{3\pi} \le \frac{1}{y}.$$

These follow from finite Taylor bounds and $\cot y \le 1/y - y/3$; the latter follows by differentiating $(1 - y^2/3)\sin y - y\cos y$. Also $\rho \ge 0$. Let

$$A = \xi^2/2 + \xi/3 + \xi^3/6, \qquad A_\ell = \xi^2/2 + \xi/2 + \xi^3/6,$$
$$B = (\xi^2/2 + \xi/3)/\pi, \qquad C = \xi^4/24 + \xi^3/18 + \xi^5/120, \qquad D = \xi^5/(360\pi).$$

Taylor inequalities give

$$\begin{aligned} L(y) = 1 + \xi - y/\pi - A_\ell y^2 &\le k_\xi(y/\pi), \\ k_\xi(y/\pi) \le U(y) &= 1 + \xi - y/\pi - Ay^2 + By^3 + Cy^4 + Dy^7, \\ 0 \le k_\xi(y/\pi) &\le 1 + \xi. \end{aligned}$$

For the lower bound, split according to the sign of $1/y - y/2$ before multiplying a sine bound. This avoids an incorrect multiplication of an inequality by a negative coefficient.

Rescale $s = \pi u/a$ and let $S = \min(\pi/(2b), \pi/a) \geq 1/a$. On the primary peak,

$$|\cos(bs)|^{1/b^2} \leq g(s)e^{-b^2 s^4/12}.$$

Indeed, successive differentiation gives $\tan z \geq z + z^3/3$, hence $-\log\cos z \geq z^2/2 + z^4/12$ before $\pi/2$. Since $e^{-v} \leq 1 - v + v^2/2$ for $v \geq 0$, the primary integrand is bounded by

$$g(s)\left[U(as) - \frac{b^2 s^4}{12}L(as) + \frac{b^4 s^8}{288}(1+\xi)\right].$$

Use $M_j = \int_0^\infty s^j g(s)\,ds$, with $M_0 = M_2 = \pi\varphi_0$, $M_1 = 1$, and $M_{j+2} = (j+1)M_j$. Integrating and extending to infinity yields

$$\begin{aligned} G_{\text{prim}} \leq{}& \varphi_0 a(1+\xi) - \frac{a^2}{\pi^2} + \varphi_0 a^3\left[-A - \frac{r^2(1+\xi)}{4}\right] \\ &+ \frac{a^4}{\pi^2}\left[\xi^2 + \frac{2\xi}{3} + \frac{2r^2}{3}\right] + \varphi_0 a^5\left[3C + \frac{5}{4}A_\ell r^2 + \frac{35}{96}(1+\xi)r^4\right] \\ &+ \frac{2\xi^5}{15\pi^2}a^8 + 128a^8. \end{aligned} \tag{8}$$

The extension error is explicit. Only the three negative monomials can increase the upper bound when extending. For $s \geq S \geq 1/a$, $1 \leq (as)^6$, so their total is at most

$$\frac{48}{\pi^2}a^8 + \left(105A\varphi_0 + \frac{945}{12}(1+\xi)\varphi_0\right)a^9 \leq 66a^8 < 128a^8.$$

The Gaussian-only primary bound is (8) with all r terms omitted. All these are actual integral comparisons, not asymptotic expansions.

4.2 The reflected peak

If $r \leq 1/2$, no reflected peak occurs. Otherwise write

$$d = \pi/b - \pi/a \geq 0, \quad w = \pi/b - s, \quad t = 1 - u = a(w-d)/\pi, \quad \beta = \pi\xi.$$

On the interior of this peak $0 < t < 1/2$ and the profile is at most $g(w)$. For $0 < t \leq 1/2$, a finite Taylor argument gives

$$0 \leq \rho(1-t) \leq \frac{\pi}{3}t^2 + t^4.$$

Here are the details. Put $z = \pi t$ and $Q = 1 - z^2/3 - z^4/32$. For $Q \geq 0$, the sextic cosine lower bound and quintic sine upper bound give

$$\begin{aligned} z\cos z - Q\sin z &\geq z\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720}\right) - Q\left(z - \frac{z^3}{6} + \frac{z^5}{120}\right) \\ &= z^5\left(\frac{13}{1440} - \frac{11z^2}{2880} + \frac{z^4}{3840}\right) \geq 0. \end{aligned}$$

Indeed, $0 \le z^2 \le 5/2$ and the parenthesis is at least $17/15360$ on that interval. For $Q < 0$, the same conclusion follows directly from $\sin z, \cos z \ge 0$. Dividing by $\sin z > 0$ yields $1 - z\cot z \le z^2/3 + z^4/32$. The reflected identity $\rho(1-t) = (1 - z\cot z)/\pi$ and $\pi^3 \le 32$ now prove the displayed bound. These comparisons are used only away from the raw-division endpoint $t = 0$; the integral ignores that single point. Using $0 \le \beta \le \pi/2$ and trigonometric addition,

$$k_\xi(1-t) \le t\cos\beta + (\beta + \pi/3)t^2 \sin\beta + t^4 \sin\beta.$$

Set $M_j(d) = \int_d^\infty (w-d)^j g(w)\,dw$. The reflected contribution is at most

$$\frac{a^2}{\pi^2}\cos\beta M_1(d) + \frac{a^3}{\pi^2}(\xi + 1/3)\sin\beta M_2(d) + \frac{a^5}{\pi^5}\sin\beta M_4(d). \tag{9}$$

At $d = 0$, the moments are 1, $\pi\varphi_0$, and $3\pi\varphi_0$. For $d \ge 0$, $g(d+t) \le g(d)g(t)$ gives $M_j(d) \le g(d)M_j(0)$.

4.3 Closing both coefficient-profile regimes

The target has the lower bound

$$c(a) + c(\xi a) \ge \varphi_0 a(1+\xi) - \frac{\varphi_0 a^3}{6}(1+\xi^3).$$

If $r \ge 19/20$, combine (8) and (9), use $\sin(\pi\xi) \le \pi\xi$, and discard $-a^2(1-\cos(\pi\xi))/\pi^2 \le 0$. The cubic coefficient is at most

$$\varphi_0\left[\frac{1}{6} + \frac{\xi^2}{2} - \frac{361(1+\xi)}{1600}\right] \le -\frac{449}{9600}\varphi_0.$$

The remaining positive terms give

$$\begin{aligned} G - c(a) - c(\xi a) &\le -\frac{449}{9600}\varphi_0 a^3 + \frac{5}{4\pi^2}a^4 + \varphi_0\left(\frac{1371}{1280} + \frac{3}{\pi^4}\right)a^5 + \left(128 + \frac{1}{240\pi^2}\right)a^8 \\ &\le -\frac{a^3}{200}. \end{aligned}$$

The last inequality is rational arithmetic using the displayed bounds on π, φ_0, a.

If $r \le 19/20$ and a reflected peak exists, then $d \ge \pi/(19a) > 2$, so $g(d) \le 1/3$. Use the Gaussian-only primary bound and (9) to obtain

$$\begin{aligned} G - c(a) - c(\xi a) &\le -\frac{2a^2}{3\pi^2} + \varphi_0 a^3\left(\frac{1}{6} + \frac{5}{18\pi}\right) + \frac{7}{12\pi^2}a^4 + \varphi_0\left(\frac{113}{3840} + \frac{1}{\pi^4}\right)a^5 + \left(128 + \frac{1}{240\pi^2}\right)a^8 \\ &\le -\frac{a^2}{20}. \end{aligned}$$

If there is no reflected peak, the same upper estimate remains valid. This proves (7). Combining with (5) and (4) proves Theorem 1.1 for $0 \le x \le a/2$.

5 Larger thresholds: the actual envelope integral

The characteristic-function envelopes below are those of Ivanisvili's Lemma 3.3 [2], and the argument follows the template of his Section 8.

Suppose $x \geq a/2$ and write $\xi = x/a$. Define

$$D(x) = \int_0^\infty \sin(sx)g(s)\,ds, \quad M(x) = \int_0^\infty s\cos(sx)g(s)\,ds, \quad N(x) = \int_0^\infty s^2\sin(sx)g(s)\,ds.$$

Integration by parts gives

$$M = 1 - xD, \qquad N = D + xM.$$

We use $D(x) \leq x$, $|M(x)| \leq 1$, $|N(x)| \leq \pi\varphi_0$, and, for $0 \leq x \leq 1$, $N(x) \leq 2x$. All follow directly from the integrals and the moment identities above.

5.1 Decomposition and the Gaussian baseline

Put $e_0 = g(1/a)$ and

$$\begin{aligned}
E_a(x) &= \frac{a}{\pi}\int_0^{\pi/a} k_\xi(as/\pi)g(s)\,ds,\\
P_a(\xi) &= \frac{a}{\pi}\int_0^{1/a} \max\{-k_\xi(as/\pi), 0\}\,[g(s) - \cos(as)^{1/a^2}]\,ds,\\
A_a(\xi) &= \frac{a}{\pi}\int_{\theta/a}^{\pi/a} |k_\xi(as/\pi)|\,|\cos(as)|^{1/a^2}\,ds.
\end{aligned}$$

The exact envelope definition (6) implies

$$J_a(x) \leq E_a(x) + P_a(\xi) + A_a(\xi) + 6e_0. \tag{10}$$

Here is the full remainder accounting. Split the u integral at $1/\pi$ and θ/π. On the first part, the envelope minus Gaussian integrand is exactly the primary loss after rescaling. On the middle part it is at most $4e_0$: $|k_\xi(u)| \leq 1 + 1/(\pi u) \leq 2$, and the relevant envelopes and Gaussian are bounded in magnitude by e_0. The same facts on the last part give the alias integrand plus $2e_0$. Both interval lengths are at most 1. The middle negative envelope bound uses $a \leq 1/2$ and $(1-a^2)\pi^2/8 \geq 1/2$.

For the baseline, write $q(z) = \cot z - 1/z + z/3$. Then

$$\rho(as/\pi) = \frac{1}{as} - \frac{as}{3} + \frac{a^2s^2}{3\pi} + (1 - as/\pi)q(as).$$

The Gaussian Fourier identities, with the scaling used here, are

$$\begin{aligned}
\int_0^\infty \cos(sx)g(s)\,ds &= \pi\varphi(x),\\
\int_0^\infty s\sin(sx)g(s)\,ds &= \pi x\varphi(x),\\
\int_0^\infty \frac{\sin(sx)}{s}g(s)\,ds &= \pi c(x).
\end{aligned}$$

The last formula follows by differentiating under the integral with a Gaussian dominator and integrating the first formula from zero to x. These identities yield the infinite baseline

$$B_a(x) = c(x) + a\varphi(x) - \frac{a^2}{\pi^2}M(x) - \frac{a^2x\varphi(x)}{3} + \frac{a^3}{3\pi^2}N(x). \tag{11}$$

In particular, the $x\varphi(x)$ term has a negative sign. Finite Taylor estimates give $q \le 0$ on $(0,\pi)$ and $|q(z)| \le z^3/30$ on $(0,1]$. For the rest, $|(1-u)q(\pi u)| \le \pi u/2$ follows from $\rho(u) \ge 1/(\pi u) - \pi u/2$ and $q \le 0$. Splitting at $s = 1/a$, the absolute cotangent remainder is at most

$$\frac{a^4}{15\pi} + \frac{a^2}{2\pi} e_0.$$

The Gaussian truncation from π/a to infinity contributes at most

$$a^2 \left(\frac{2}{\pi^2} + \frac{1}{\pi^3} + \frac{2}{3\pi} + \frac{a^2}{3\pi^3} \right) g(\pi/a) \le e_0.$$

This follows from $1/(as) \le 1/\pi$ on that tail and the moment estimates $\int_R^\infty s g(s)ds = g(R)$, $\int_R^\infty g(s)ds \le g(R)/R$, and $\int_R^\infty s^2 g(s)ds \le (R + 1/R)g(R)$. Indeed, the displayed parenthesis is at most 4 for $a \le 1$ and $\pi \ge 1$, while $4a^2 \le 1$ for $a \le 1/2$, and $g(\pi/a) \le e_0$. Consequently

$$E_a(x) \le B_a(x) + \frac{a^4}{15\pi} + 2e_0. \tag{12}$$

5.2 Two primary-loss estimates

For $0 \le z \le 1$,

$$0 \le -\log\cos z - z^2/2 \le z^4/12 + z^6/20.$$

For the upper bound, the finite sine and cosine polynomials show $\tan z \le z + z^3/3 + 3z^5/10$; integration proves the assertion. Thus for $0 \le s \le 1/a$,

$$0 \le g(s) - \cos(as)^{1/a^2} \le g(s) \left(\frac{a^2 s^4}{12} + \frac{a^4 s^6}{20} \right).$$

To control the cotangent correction globally, write $h(u) = \rho(u) - 1/(\pi u)$. Since $-\pi u/2 \le h(u) \le 0$ and $\sin(sx) \le sx$ for $x, s \ge 0$,

$$h(u)\sin(sx) \ge h(u)sx \ge -axs^2/2, \qquad u = as/\pi.$$

This estimate holds even when the sine is negative. Together with the global sine and cosine Taylor bounds, it gives, for $0 < s < 1/a$,

$$k_\xi(as/\pi) \ge 1 - as/\pi + \xi - s^2 \left(\frac{x^2}{2} + \frac{x^3}{6a} + \frac{ax}{2} \right).$$

Integrate the resulting negative-part bound to get

$$P_a(\xi) \le \varphi_0 \left(\frac{5}{4}a^3 + \frac{21}{4}a^5 \right) \left(\frac{x^2}{2} + \frac{x^3}{6a} + \frac{ax}{2} \right). \tag{13}$$

Alternatively, use $|k_\xi(as/\pi)| \le 1 + 1/(as)$ to obtain

$$P_a(\xi) \le \frac{\varphi_0 a^3}{4} + \frac{a^2}{6\pi} + \frac{3\varphi_0 a^5}{4} + \frac{2a^4}{5\pi}. \tag{14}$$

There is no extra tail in these definitions: the middle-frequency errors were already included in (10).

5.3 A compensated alias estimate for every phase

Let $t = 1 - u$ and $\beta = \pi\xi$. On the alias interval $0 < t < 1/2$,

$$\rho(1-t) \le \frac{\pi}{3}t^2 + t^4 \le \frac{4}{3}t^2.$$

Trigonometric addition, $|\sin(\beta t)| \le \beta t$, and $|\cos(\beta t)| \le 1$ imply

$$|k_\xi(1-t)| \le t|\cos\beta| + (\pi\xi + 4/3)t^2|\sin\beta| + \frac{4\pi}{3}\xi t^3.$$

The reflected profile is at most $\exp(-\pi^2 t^2/(2a^2))$. Extending the nonnegative Gaussian integrals gives the actual alias bound

$$A_a(\xi) \le \frac{a^2}{\pi^2}|\cos\beta| + \frac{\varphi_0}{\pi}(\xi+\delta)a^3|\sin\beta| + \frac{8\xi}{3\pi^3}a^4, \qquad \delta = \frac{4}{3\pi}. \tag{15}$$

For $Q > 0, K \ge 0$, Cauchy–Schwarz and a rationalized square root give

$$Q|\cos\beta| + K|\sin\beta| - Q \le \frac{K^2}{2Q}.$$

The formal proof uses the corresponding squared polynomial identity and positivity, so no square-root approximation is needed. This elementary oscillation inequality is also Ivanisvili's Lemma 8.6 in the lower-half analysis [2]; neither the inequality nor the general phase-compensation idea is claimed as new. Here it is combined with the upper-kernel estimate (15) and the uniform bounds needed for the previously unresolved upper half. Set $\eta = 1/2 + 3/\pi^2$. The exact coefficient comparison

$$\frac{\varphi_0^2}{2}(\xi+\delta)^2 + \frac{8\xi}{3\pi^3} \le \frac{\varphi_0^2}{2}(\xi+\eta)^2 + \frac{6}{\pi^4}(\xi+1) \quad (\xi \ge 0)$$

follows from $\delta \le 4/9$, $\eta \ge 4/5$, $\varphi_0^2 \ge 1521/10000$, $\pi^4 \le 100$, and $\pi^3 \ge 27$. For example, the linear coefficient on the right minus the left is nonnegative since $\varphi_0^2(\eta-\delta) + 6/\pi^4 \ge (1521/10000)(16/45) + 3/50 > 8/81$. The constant coefficient is also nonnegative. Hence

$$A_a(\xi) - \frac{a^2}{\pi^2} \le \frac{\varphi_0^2}{2}(\xi+\eta)^2a^4 + \frac{6}{\pi^4}(\xi+1)a^4. \tag{16}$$

This cancels the leading $-a^2/\pi^2$ in (11); bounding the alias by its absolute magnitude before preserving that cancellation would lose the needed precision.

5.4 The complete master estimate

The exponential series implies $e_0 \le 6!(2a^2)^6 = 46080a^{12}$. For $a \le 7/100$, exact rational arithmetic gives

$$9e_0 \le a^4/1000, \qquad \frac{a^4}{15\pi} + 9e_0 \le a^4/40.$$

The preceding decomposition and baseline require at most $8e_0$; we use the slightly larger $9e_0$ allowance uniformly. Combine (10), (11), (12), $M = 1 - xD$, and (16). For either primary bound P_{bd} in (13)–(14),

$$\begin{aligned} J_a(x) - c(x) - c(a) \le a\varphi(x) - c(a) - \frac{a^2x\varphi(x)}{3} + \frac{a^2xD(x)}{\pi^2} + \frac{a^3N(x)}{3\pi^2} + P_{\mathrm{bd}} \\ + \frac{\varphi_0^2}{2}(\xi+\eta)^2a^4 + \frac{6}{\pi^4}(\xi+1)a^4 + \frac{a^4}{40}. \end{aligned} \tag{17}$$

5.5 Closing the intermediate thresholds

Use $D \le x$, $N \le 2x$, $\varphi_0(1-x^2/2) \le \varphi(x) \le \varphi_0(1-x^2/2+x^4/8)$, and $c(a) \ge \varphi_0(a-a^3/6)$. With $K = 32/25$, the right side of (17), using (13), divided by $\varphi_0 a^3$ is at most

$$\mathcal{P} = \frac{1}{6} - \frac{\xi^2}{2} - \frac{\xi}{3} + a^2\left[\frac{\xi^4}{8} + \frac{\xi^3}{6} + K\left(\frac{\xi^2}{2} + \frac{\xi^3}{6} + \frac{\xi}{2}\right)\right] + a(C_2\xi^2 + C_1\xi + C_0),$$
$$C_2 = \frac{100}{351} + \frac{1}{5}, \quad C_1 = \frac{400}{1053} + \frac{1}{3}, \quad C_0 = \frac{5}{36} + \frac{200}{1053} + \frac{5}{78}.$$

These constants dominate the coefficients using the earlier rational bounds on π, φ_0 and $\eta \le 5/6$. For $1/2 \le \xi \le 1$, use $\xi^4 \le \xi^2$ and $\xi^3 \le \xi^2$ in the K term, and $\xi^3 \le \xi$ in the standalone cubic term. Set $\alpha = 7/100$ only in nonnegative coefficients. The exact checks are

$$\frac{1}{6} + \alpha C_0 \le \frac{1}{5},$$
$$-\frac{1}{2} + \alpha C_2 + \alpha^2\left(\frac{1}{8} + \frac{2K}{3}\right) \le -\frac{9}{20},$$
$$-\frac{1}{3} + \alpha C_1 + \alpha^2\left(\frac{1}{6} + \frac{K}{2}\right) \le -\frac{11}{40}.$$

They give

$$\mathcal{P} \le \frac{1}{5} - \frac{9}{20}\xi^2 - \frac{11}{40}\xi \le -\frac{1}{20}.$$

For $\xi \ge 1$ and $a\xi = x \le 1$, use $a^2\xi^4 \le \xi^2$, $a^2\xi^3 \le \xi$, and $a^2\xi^3 \le a\xi^2$ in the K term. The remaining coefficient checks are

$$-\frac{3}{8} + \alpha C_2 + K\left(\frac{\alpha^2}{2} + \frac{\alpha}{6}\right) \le -\frac{3}{10},$$
$$-\frac{1}{6} + \alpha C_1 + \frac{K\alpha^2}{2} \le -\frac{1}{10}.$$

Thus

$$\mathcal{P} \le \frac{1}{5} - \frac{3}{10}\xi^2 - \frac{1}{10}\xi \le -\frac{1}{5}.$$

Therefore

$$J_a(x) - c(x) - c(a) \le -\varphi_0 a^3/20 \qquad (a/2 \le x \le 1). \tag{18}$$

5.6 Closing the large thresholds before the tail cutoff

Use $xD = 1 - M \le 2$, $|N| \le \pi\varphi_0$, $\varphi(x) \le \varphi(1)$, and (14); discard the favorable negative $x\varphi(x)$ term. Since $e^{-1/2} \le 5/8$, $\varphi(1) - \varphi_0 \le -117/800$. Divide (17) by a. Using $ax^2 \le 1$ and $a^2x \le a$, all positive terms are bounded by

$$R(a) = \frac{a^2}{15} + \frac{2a}{9} + \frac{2a^2}{45} + \left(\frac{a^2}{10} + \frac{a}{18} + \frac{3a^4}{10} + \frac{2a^3}{15}\right)$$
$$+ \frac{2}{25}\left(1 + \frac{5a}{3} + \frac{25a^3}{36}\right) + \frac{2}{27}(a + a^3) + \frac{a^3}{40}.$$

Every coefficient is nonnegative, and

$$R(7/100) = \frac{1553895653}{13500000000}, \qquad -\frac{117}{800} + R(7/100) < -\frac{1}{40}.$$

Thus

$$J_a(x) - c(x) - c(a) \le -a/40 \qquad (x \ge 1,\ ax^2 \le 1). \tag{19}$$

5.7 The far tail and completion

If $ax^2 \ge 1$, then $x \ge 1$ and $c(a) \ge a/3$. Since $a \le 7/100 < 1/12$, the cubic term of the exponential series gives

$$e^{-1/(2a)} \le 48a^3 \le a/3.$$

The Gaussian Mills bound therefore implies

$$1 - \Phi(x) \le \varphi(x)/x \le e^{-1/(2a)} \le c(a).$$

Consequently $H(x) + H(a) \ge 1$, so (1) follows from the actual probability being at most 1. This branch makes no assertion that the smoothing envelope itself is below the target in the far tail.

The small-threshold profile bound, (18), (19), and the far-tail probability bound cover every $x \ge 0$, including all range joins. Together with (4)–(6), they prove Theorem 1.1.

6 Formal verification and reproducibility

Fresh replays on 30 October 2026 compiled all 62 modules from source into new object directories against a Mathlib checkout built from source at the pinned revision, once with the default import trust level and once with `-trust=0`, which also rechecks imported declarations. Both passed, and all 14 audited declarations depend only on `propext`, `Classical.choice` and `Quot.sound`. Their records are in `verification/`.

The theorem names in `KellerKlein/Main.lean` are `small_coefficient_upper_concentration`, `small_coefficient_upper_concentration_normal`, and `small_coefficient_upper_cdf`. The latter two use `ProbabilityTheory.cdf (ProbabilityTheory.gaussianReal 0 1)` directly. The finite law is the cardinality of the appropriate subset of $\{\text{false}, \text{true}\}^{\mathrm{Fin}\, n}$ divided by 2^n. Coordinate-wise Boolean sign flips prove equality in law under $v_i \mapsto |v_i|$. The exact characteristic function is proved by factoring the finite sum; no independence or distributional assertion is an extra hypothesis.

The actual theorem hypotheses are exactly those in Theorem 1.1. In particular, the formal result has no hypothesis named “master bound”, “Prawitz bound”, “envelope validity”, or “Gaussian approximation”. Every analytic fact used in those reductions is proved in the local dependency chain or comes from pinned Mathlib theorems.

The clean verifier copies every local proof into a new source directory and builds entirely fresh project objects with `-trust=1 -j1 -DautoImplicit=false -Ddebug.skipKernelTC=false`. Every newly compiled declaration is kernel-checked; the positive import trust level permits reuse of previously compiled library declarations. The verifier checks every dependency’s Git commit and clean tracked worktree, rejects unsupported proof mechanisms, and checks printed axiom dependencies of the actual central theorems. It also checks that the source files did not change during verification. Mathlib’s compiled dependency objects are reused read-only. The Git checks establish the dependency source

revisions; with a downloaded cache, the correspondence of the compiled objects to those sources is a trusted input. The verifier also accepts `-trust 0`, which additionally rechecks imported declarations.

For an independent installation on Linux or WSL with Elan, enter the package's `lean/` directory and run:

```
lake exe cache get
python3 verify.py --lean lean --mathlib .lake/packages/mathlib
```

Keep the included manifest; do not run a dependency update before reproducing. The verifier writes three reports into a `verification/` directory next to `verify.py`: `build.log`, `axioms.txt`, and `certificate.json`. The certificate records source hashes, dependency commits, compiler identity, module exit codes, and central axiom dependencies.

Three small Prawitz-support modules are ported from Jiyuan-Tan/CausalSmith, at the following commit, under Apache 2.0:

`a7085d88d2e4b30003db7097ca374f4718da0f01`.

The original files, license, notice and port descriptions are in `third_party/causalsmith`. The recurrence approach was also informed by the publicly available CoolRmal/BerryEsseen development, at commit

`bfb22608ad18b4fa9fda8dd9a1ed2c2b6fbc29e7`.

Its code was not copied. The exact probability reduction and the analytic completion are local proofs. Their mathematical ingredients are not all claimed to be novel.

6.1 The external prior results

The full-conjecture corollary rests on Ivanisvili's Theorems 1.1 and 10.9, whose proofs combine analysis with interval-arithmetic certificates. A separate replay of those certificates was carried out during this research, at 96-bit precision and with an added guard for a rounding issue in the selection of profile maxima; it reported no failure. That replay is not part of this package, and it does not bring the external results inside the Lean trust boundary.

7 Novelty assessment

The searches used the conjecture name and alternative formulations: Rademacher Kolmogorov distance, the $\Phi(a) - 1/2$ bound, closed symmetric concentration, the small maximum-coefficient regime, and the corresponding one-sided discrepancies. Primary journal and arXiv sources, recent preprints, author repositories, thesis references, MathOverflow searches, and mathematical database entries were checked. The relevant latest primary source found is Ivanisvili's October 2026 manuscript; its stated unresolved range matches the new theorem exactly.

The search also tested closely related general concentration theorems rather than relying only on the conjecture's name. Mattner's sharp common-threshold binomial comparison [7] does not directly imply the new bound. For example, take 400 coefficients equal to $3/100$ and 400 equal to $4/100$; their squared sum is 1. At $x = 1$, the best application of that comparison over its threshold parameter is greater than $3/4$, whereas $H(1) + H(1/25) < 539/750 < 3/4$.

Older sharp asymptotic bounds were checked as a separate possible source of a hidden implication. Put $\ell = \sum_i |v_i|^3$. In the Rademacher case, the two leading terms of Shevtsova's symmetric-sum estimate [8] combine to $\varphi_0 \ell$, with a positive remainder; Chistyakov's symmetric consequence [9] likewise adds a positive remainder to $\varphi_0 \ell$. For equal magnitudes, $\ell = a$, while

$$c(a) = \varphi_0 a - \frac{\varphi_0}{6} a^3 + O(a^5) < \varphi_0 a.$$

The inspected estimates therefore do not supply the required signed cubic improvement, and this obstruction persists for nearby unequal vectors.

Equal-weight results and point-mass bounds are prior mathematics and are excluded from the originality claim. The proposed new content is the uniform upper bound for arbitrary unequal coefficients with $a \le 0.07$.

The contribution's significance does not rest on its formalization alone. It supplies an infinite coefficient regime and all thresholds at the exact conjectured radius, and closes the remaining regime relative to the latest reported result. Formalization is an additional check on that mathematical claim. Whether the combined resolution merits a particular publication venue is for independent specialists to assess.

The following limits remain explicit.

- Novelty is supported by the located public literature, not certified against unpublished work or simultaneous research. The frontier moved only nine days before this investigation; priority therefore remains provisional.
- The full-conjecture corollary uses the external computer-assisted results, which are unrefereed and outside the Lean trust boundary.
- The Lean verification relies on the pinned Lean compiler and on the compiled Mathlib objects. It does not verify the compiler implementation or the host operating system.

Numerically, the bound is tightest for even equal weights at $x = 0$, where the relative slack $1 - \sup_x(\mathbb{P}(X \le x) - \Phi(x))/c(a)$ is about $a^2/12$; the upper half is therefore not attained in the range $a \le 7/100$.

8 Research process and AI disclosure

This is fully autonomous agentic research. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed, no expert validation is claimed, and the novelty assessment is provisional.

Autonomous AI research agents produced the proof, the Lean formalization and an earlier version of this note in an earlier session. The records of that session state that the note was prepared with ChatGPT; the model version is not recorded.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It coordinated an independent AI referee check, which re-derived the main chain of estimates by hand, compared the theorem with Ivanisvili's manuscript and with the Lean statements, recomputed exact sign-vector counts for several hundred coefficient vectors (comparing them with floating-point values of Φ) and found no violation, and ran the exact rational constant checks. It ran the fresh Lean replays

described in Section 6 and revised this manuscript: it made the dependence of the full conjecture on unrefereed external results explicit, credited the strategy and lemmas taken from Ivanisvili's manuscript, and removed claims that rested on files not included here. On 1 November 2026 it also revised the presentation (front matter, abstract, introduction and this section) without changing the mathematics. AI referee checks are not human peer review.

Lean verification establishes only the formal statements described in Section 6. The remaining limits, including the conditional status of the full conjecture, are listed in Section 7.

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