The sharp level-one Fourier weight at density one eighth

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Abstract

The level-one Fourier weight of a Boolean function is the total squared correlation of the function with the individual coordinates. Yu asks for the largest level-one weight of a Boolean function of mean 1/8 (Problem 2 in Section VII of [4]). This note proves that the maximum is 3/64 in every dimension at least three, attained by the indicator of a codimension-three subcube. Equivalently, a subset of the Boolean cube with density 1/8 has average Hamming distance at least $(n-3)/2$. Fang and Huang proved this bound first, together with a characterization of the equality cases [2]; the present proof is independent, and no priority is claimed (see the dated note below). The proof establishes a sharp positive-part inequality for Rademacher sums. An even polynomial of degree eighteen bounds the positive part, and an exact symmetric moment certificate, with rational coefficients and nonnegative terms, bounds the expectation of that polynomial in every dimension. A companion Lean formalization proves the moment identities, the Fourier and distance bounds in every dimension, and their sharpness witnesses.

Priority and subsequent correspondence

Added 30 October 2026. This note was first published in this repository on 27 October 2026. On 30 October 2026, Professor Lei Yu informed me that Yijia Fang and Hao Huang had already proved the same density-1/8 Fourier-weight bound, and that their result also characterizes the equality cases. He told me that their manuscript had been shared privately approximately one month earlier; I have not seen that version. Their proof was submitted to arXiv on 29 October 2026 [2]. I was previously unaware of their work, and the novelty searches carried out by my research agents had not found it. I confirmed the public arXiv record, whose summary states the same bound in its average-distance form together with the equality cases; I have not compared the two proofs. Fang and Huang proved the bound before I did. I acknowledge their earlier work and make no originality or priority claim for the bound. I have withdrawn the statements of the original version that presented the bound as the contribution of this note. The proof, its certificates and its formalization are unchanged; they remain available as an independent derivation and formal check.

1 Introduction

Let $X = (X_1, \dots, X_n)$ be uniform on $\{-1,1\}^n$, and let $f : \{-1,1\}^n \to \{0,1\}$. Write

$$a = \mathbb{E}f(X), \qquad b_i = \mathbb{E}[X_i f(X)], \qquad W_1(f) = \sum_{i=1}^{n} b_i^2.$$

Let $W(1/8)$ be the dimension-free supremum of $W_1(f)$ over all such f with $a = 1/8$. Yu asks for the value of $W(1/8)$ in Problem 2 of Section VII of [4]. Section III-C of that paper gives an upper bound of approximately 0.0495142, and the codimension-three subcube gives the lower bound $3/64 = 0.046875$. The following theorem shows that the lower bound is the answer.

Theorem 1. *The following bound holds:*

$$\boxed{a = \frac{1}{8} \quad \Longrightarrow \quad W_1(f) \le \frac{3}{64}.} \tag{1}$$

For every $n \ge 3$, equality is attained by the indicator of the subcube $X_1 = X_2 = X_3 = 1$. In the average-distance formulation, if $A \subseteq \{-1, 1\}^n$ and $|A| = 2^{n-3}$, then

$$\boxed{\frac{1}{|A|^2} \sum_{x,y \in A} d_H(x, y) \ge \frac{n-3}{2}.} \tag{2}$$

Equality holds in (2) for a codimension-three subcube.

Consequently the maximum of W_1 at mean $1/8$ is exactly $3/64$ in every admissible dimension, and $W(1/8) = 3/64$. Pairs in (2) are ordered, sampled independently, and may coincide.

The result concerns exactly the density $1/8$. It does not determine the optimum at every density or the full noise stability of a subcube. No classification of all equality cases is claimed in this note; Fang and Huang [2] give one.

Theorem 1 was first proved by Fang and Huang [2]. What this note offers is an independent proof through an explicit rational certificate (a polynomial majorant combined with a dimension-independent moment identity) and a Lean formalization of that proof; the techniques themselves are established (Section 2). The Lean development (Section 8) proves the polynomial and moment identities, their interpretation on the uniform cube in every dimension, inequalities (1)–(3) and the sharpness witnesses. A separate Python checker rechecks the certificate in exact rational arithmetic. The relation to Yu's problem and to earlier bounds rests on the cited literature.

Section 3 reduces the theorem to an inequality for Rademacher sums, Sections 4 and 5 give its two certificates, and Sections 6 and 7 derive the Fourier and distance bounds.

2 Relation to previous bounds

The minimum-average-distance problem in Hamming space goes back to Ahlswede and Katona [1]. Fu, Wei, and Yeung obtained linear-programming bounds and solved several cases, including quarter-cube size [3]. Yu and Tan later characterized the asymptotically best bounds from that linear program [5]. In the Fourier formulation, Yu [4] gives the upper bound quoted above for $W(1/8)$.

Fang and Huang [2] closed the gap between $3/64$ and Yu's upper bound before this note, proving that the subcube value is an upper bound and characterizing the equality cases. This note gives an independent proof of the bound.

The proof combines a polynomial majorant with a dimension-independent moment identity. Its coefficients are rational. Nonnegative Bernstein coefficients establish the one-variable inequalities, and a sum of nonnegative symmetric terms establishes the moment inequality. These are established methods. The bound itself is due to Fang and Huang [2]; this note gives a proof through an explicit certificate, with a Lean formalization, and makes no originality or priority claim for the bound.

3 The auxiliary sharp Rademacher inequality

For any real coefficients $c_1, \dots, c_n$ with $\sum_i c_i^2 = 1$, put

$$S = \sum_i c_i X_i.$$

The proof rests on the following stronger auxiliary estimate:

$$\mathbb{E}(|\sqrt{3}S| - 9/4)_+ \le 3/16, \qquad t_+ = \max(0, t). \tag{3}$$

This constant is attained when $c_1 = c_2 = c_3 = 1/\sqrt{3}$ and all other coefficients vanish: $|\sqrt{3}S|$ is 3 with probability 1/4 and 1 with probability 3/4.

The proof uses a degree-eighteen even polynomial

$$P(t) = \sum_{j=0}^{9} p_j t^{2j} \tag{4}$$

with rational coefficients, given in Appendix A. The two properties required are

$$P(t) \ge \max(0, |t| - 9/4) \quad \text{for all } t \in \mathbb{R}, \tag{5}$$

and

$$\mathbb{E}P(\sqrt{3}S) \le 3/16 \quad \text{whenever } \sum_i c_i^2 = 1. \tag{6}$$

The exact contacts $P(1) = 0$ and $P(3) = 3/4$ explain why equality can occur at the three-coordinate Rademacher sum. Contacts alone do not prove either global inequality; Sections 4 and 5 provide their certificates.

4 Global verification of the one-variable majorant

Set $p(u) = \sum_{j=0}^{9} p_j u^j$. Exact polynomial division gives

$$p(u) = (u-1)^2 R(u), \qquad \deg R = 7, \tag{7}$$

and

$$P(t) - t + 9/4 = (t-3)^2 T(t), \qquad \deg T = 16. \tag{8}$$

Both R and T are nonnegative on $[0, \infty)$. We now give a finite rational certificate of this assertion and explain how it is checked.

For a degree-d polynomial Q on a rational interval $[L, H]$, use the Bernstein representation

$$Q(x) = \sum_{i=0}^{d} \beta_i \binom{d}{i} \left(\frac{x-L}{H-L}\right)^i \left(\frac{H-x}{H-L}\right)^{d-i}. \tag{9}$$

Every basis term is nonnegative on the interval. Thus nonnegative rational β_i prove $Q \ge 0$ there. For the remaining ray, a shifted expansion $Q(B+s) = \sum_i \gamma_i s^i$ with $\gamma_i \ge 0$ proves nonnegativity for $s \ge 0$.

The complete exact R, T, all Bernstein coefficient arrays, and both shifted-tail arrays are in `univariate_certificate.json`. The intervals are:

- R: [0,32], [32,64], [64,80], [80,96], [96,128]; tail begins at 128.
- T: [0,4], [4,5], [5,11/2], [11/2,6], [6,7], [7,8], [8,9], [9,19/2], [19/2,10], [10,12], [12,16]; tail begins at 16.

These are contiguous covers of $[0, 128]$ and $[0, 16]$, respectively. All their Bernstein coefficients and all shifted-tail coefficients are nonnegative rational numbers. Every identity (7)–(9) and every tail identity is explicitly written in the Univariate section of `SharpEighth.lean`, checked by Lean's `ring` tactic, and combined with elementary order reasoning. This proves $R(u) \geq 0$ for $u \geq 0$ and $T(t) \geq 0$ for $t \geq 0$ without approximation.

Equation (7), applied to $u = t^2$, proves $P(t) \geq 0$ for every real t. Equation (8) proves $P(t) \geq t - 9/4$ for $t \geq 0$. Since P is even, apply the latter inequality to $|t|$ to obtain (5).

5 The dimension-independent moment certificate

5.1 Ordered orbit sums

Let $z_i = c_i^2$ and $V = \sum_i z_i$. For a finite list of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define

$$O_\lambda(z) = \sum_{\iota : \{1,\ldots,\ell\} \hookrightarrow \{1,\ldots,n\}} \prod_{r=1}^{\ell} z_{\iota(r)}^{\lambda_r}. \tag{10}$$

The sum is over **ordered injections**. Define $O_{()} = 1$. If $\ell > n$, the sum is empty and equals zero. Permuting the list λ leaves O_λ unchanged. If λ is a partition and $\mathrm{aut}(\lambda)$ is the product of the factorials of its part multiplicities, then the ordinary monomial symmetric polynomial is $m_\lambda = O_\lambda / \mathrm{aut}(\lambda)$.

Two identities control the algebra:

$$V O_\lambda = O_{(\lambda,1)} + \sum_{r=1}^{\ell} O_{\lambda + e_r}, \tag{11}$$

and, when a new coordinate a is adjoined,

$$O_\lambda(a, z) = O_\lambda(z) + \sum_{r=1}^{\ell} a^{\lambda_r} O_{\lambda \setminus r}(z). \tag{12}$$

To prove (11), expand $V O_\lambda$. The coordinate chosen from V either occurs at the unique marked position r in the injection or is new. The first case gives $O_{\lambda + e_r}$, and the second gives $O_{(\lambda,1)}$. To prove (12), classify an injection according to whether it avoids the new coordinate or uses it at its unique position r. Deleting that position is a bijection with injections defining $O_{\lambda \setminus r}(z)$. No division by z_i is used, so zero coordinates and every dimension are covered.

5.2 Rademacher moments

For integers $0 \leq j \leq 9$, define

$$M_j(z) = \sum_{\lambda \vdash j} \frac{(2j)!}{\mathrm{aut}(\lambda) \prod_r (2\lambda_r)!} O_\lambda(z). \tag{13}$$

Then

$$M_j(c_1^2, \ldots, c_n^2) = \mathbb{E} S^{2j}. \tag{14}$$

Indeed, expand S^{2j}. Independence and symmetry kill every monomial containing an odd exponent. The surviving even exponent pattern 2λ has multinomial coefficient $(2j)! / \prod_r (2\lambda_r)!$. Passing from distinct monomials to the ordered-orbit convention divides by $\mathrm{aut}(\lambda)$.

The Lean proof uses a finite recurrence instead of assuming a general multinomial theorem. Averaging the two signs of a new coefficient gives

$$\mathbb{E}(aX_0 + S)^{2j} = \sum_{k=0}^{j} \binom{2j}{2k} a^{2k} \mathbb{E} S^{2(j-k)}. \tag{15}$$

The ten explicit tables $M_0, \dots, M_9$ in the CertificateData section satisfy the identical recurrence under (12); the ten finite coefficient equalities are kernel-checked in the MomentRecurrenceData section. In dimension zero both sides of (14) equal 1 for $j = 0$ and 0 otherwise. Induction on dimension, using (12) and (15), proves (14) for every needed $0 \le j \le 9$. Thus this step is not a check only in a fixed number of variables.

5.3 Weighted squares

For a list λ, real vector $a = (a_1, \dots, a_\ell)$, and scalar b, define

$$G_{\lambda;a,b}(z) = \sum_{\iota \text{ injective}} \left(\prod_r z_{\iota(r)}^{\lambda_r} \right) \left(\sum_r a_r z_{\iota(r)} + bV \right)^2. \tag{16}$$

Both $O_\lambda(z)$ and $G_{\lambda;a,b}(z)$ are nonnegative when all $z_i \ge 0$. Expanding the square and using (11) expresses (16) in the ordered-orbit basis. The coefficients a_r attach to the original positions of λ: the two marked exponents are incremented before sorting the resulting list. This convention matters when equal exponents occur.

The key exact identity is

$$\frac{3}{16} V^9 - \sum_{j=0}^{9} p_j 3^j M_j(z) V^{9-j} = \sum_{r=1}^{28} w_r H_r(z), \qquad w_r > 0, \tag{17}$$

where the H_r and weights are given in Appendix B. The first five terms are ordered-orbit monomials of degree nine; the other twenty-three are weighted squares with $|\lambda| = 7$. Thus every term is homogeneous of degree nine.

For clarity, (17) is a finite rational polynomial identity, independent of the numerical values of z or the dimension. Expand (16), repeatedly use (11), collect equal sorted exponent lists, and compare rational coefficients. There are exactly thirty partitions of nine. One coefficient is zero on both sides; the other twenty-nine normalized coefficients agree. `certificate_coefficient_identity` in the CertificateData section checks this equality by kernel reduction of exact rational arithmetic. The SymmetricCertificate section proves that collection, scaling, multiplication by V, and canonical sorting preserve evaluation. The OrbitBridge and SquareSemantics sections connect those operations to (10) and (16). Consequently the finite coefficient check proves (17) for every dimension.

Now take $z_i = c_i^2$, so $z_i \ge 0$ and $V = 1$. The right side of (17) is nonnegative. By (14), the left side is $3/16 - \mathbb{E}P(\sqrt{3}S)$. This proves (6). Combining (5) with (6) proves (3).

6 Proof of the sharp Fourier bound

Let $W = W_1(f)$. If $W = 0$, inequality (1) is immediate. Otherwise put $c_i = b_i/\sqrt{W}$ and $S = \sum_i c_i X_i$. Then

$$\sum_i c_i^2 = 1, \qquad \mathbb{E}[f(X)S] = \frac{\sum_i b_i^2}{\sqrt{W}} = \sqrt{W}.$$

Set $Z = \sqrt{3}S$. Its distribution is symmetric, because complementing every sign replaces Z by $-Z$. Since $9/4 \ge 0$, pointwise

$$(Z - 9/4)_+ + (-Z - 9/4)_+ = (|Z| - 9/4)_+.$$

Therefore (3) implies

$$\mathbb{E}(Z - 9/4)_+ = \frac{1}{2}\mathbb{E}(|Z| - 9/4)_+ \le 3/32. \tag{18}$$

For a Boolean-valued f, the elementary pointwise inequality

$$fZ \le \frac{9}{4}f + (Z - 9/4)_+ \tag{19}$$

holds in both cases $f = 0$ and $f = 1$. Taking expectations, using $\mathbb{E}f = 1/8$ and (18), gives

$$\sqrt{3}\sqrt{W} = \mathbb{E}[fZ] \le \frac{9}{32} + \frac{3}{32} = \frac{3}{8}.$$

Both sides are nonnegative, so squaring yields $3W \le 9/64$, which is (1).

For the subcube indicator $f = \mathbf{1}\{X_1 = X_2 = X_3 = 1\}$, the first three coefficients b_i are $1/8$, and all remaining coefficients vanish by independence and zero mean of the unused sign. Hence $W_1(f) = 3/64$, proving attainment in every $n \ge 3$.

7 Average distance

Let A have density $a > 0$ and $f = \mathbf{1}_A$. For a uniform point of A, its ith coordinate mean is b_i/a. If U, V are independent uniform points of A, then

$$\mathbb{E}d_H(U, V) = \frac{1}{2}\sum_i (1 - \mathbb{E}[U_i V_i]) = \frac{n}{2} - \frac{W_1(f)}{2a^2}. \tag{20}$$

At $a = 1/8$, substitute (1) to get (2). For a codimension-three subcube, (20) and its exact Fourier weight give equality. The Lean development defines the weighted average as the double uniform-cube average of $f(x)f(y)d_H(x, y)$, divided by $(\mathbb{E}f)^2$; for an indicator this is exactly the ordered-pair average over A.

8 Formalization and exact certificates

The companion formalization `SharpEighth.lean` uses Lean 4.19.0 and Mathlib revision `c44e0c8ee63ca166450922a373c7409c5d26b00b`. It includes the finite rational identities and their interpretation for every cube dimension. Polynomial identities use `ring`. Finite rational table equalities use `decide +kernel`. The moment recurrence then connects these identities to the uniform probability measure on the Boolean cube.

The main statements are `SharpEighth.sharp_rademacher_hinge`, which is inequality (3); `SharpEighth.sharp_eighth_fourier_bound`, which is inequality (1) for real-valued functions on `Fin n -> Bool` taking only the values 0 and 1; `SharpEighth.average_distance_eighth`, which is inequality (2) in the weighted form of Section 7; and `SharpEighth.exact_eighth_fourier_optimum` and `SharpEighth.exact_eighth_distance_optimum`, which state that $3/64$ and $(n - 3)/2$ are the attained optimum values in every dimension $n \ge 3$. The file `Audit.lean` prints the axioms on which these and the certificate statements depend; the README gives the build commands and permits only `propext`, `Classical.choice` and `Quot.sound`.

A separate Python checker, `verify_certificate.py`, uses exact rational arithmetic. It checks all thirty symmetric coefficients, all twenty-eight positive weights, sixteen Bernstein intervals, both

polynomial factorizations, both tail expansions, interval coverage, and the contact values. The README gives the commands and dependency versions. These checks use the stated certificate. They do not depend on a numerical optimization solver.

9 Research process and AI disclosure

This work is fully autonomous agentic research. As recorded for this result, the initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT, and GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities; this check is not expert validation or peer review. The priority note of 30 October 2026 records later correspondence. The editorial revision of 1 November 2026 was made with Opus 5.5 by Anthropic used with Claude Code; it changed the presentation only, not the mathematics, the certificates or the formalization.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed. Section 8 states the scope of the formal and exact checks, and Section 1 states the limits of the result.

A Exact polynomial

In $P(t) = \sum_{j=0}^{9} p_j t^{2j}$, use:

j	p_j
0	$\frac{2065204456063070436648471 09}{362500000000000000000000000000}$
1	$\frac{-195397256376749992156612335 1}{217500000000000000000000000000000}$
2	$\frac{2814044752324915893803393 47}{489375000000000000000000000000000}$
3	$\frac{7409017415686803346331558 09}{244687500000000000000000000000000}$
4	$\frac{-3247861969010612115496283 47}{978750000000000000000000000000000}$
5	$\frac{796521088493109}{50000000000000000000}$
6	$\frac{-41161541352553493}{100000000000000000000000}$
7	$\frac{29617055737416117}{500000000000000000000000000}$
8	$\frac{-5574570362723033}{125000000000000000000000000}$
9	$\frac{341703734117989}{250000000000000000000000000}$

B Exact nonnegative symmetric identity

Use (17) with the following terms. The monomial weights below are in the **ordered-orbit** convention of (10), matching the CertificateData section. In `exact_certificate.json`, monomial entries use the conventional m_λ normalization; dividing those five weights by $\mathrm{aut}(\lambda)$ gives the weights printed here. The square terms use the same ordered-injection convention in both representations.

Every numerator and denominator in the final column is positive. This table and Appendix A completely determine identity (17).

r	H_r	a	b	w_r
1	$O_{(3,1,1,1,1,1,1)}$	–	–	$\frac{16646137361744047018987645 87}{67968750000000000000000000000}$
2	$O_{(2,2,2,2,1)}$	–	–	$\frac{10898934482037153838833003 1}{45312500000000000000000000}$
3	$O_{(2,2,2,1,1,1)}$	–	–	$\frac{36368430041679577673356417 9}{11328125000000000000000000}$
4	$O_{(2,2,1,1,1,1,1)}$	–	–	$\frac{23369443801983910767596555 27}{45312500000000000000000000}$
5	$O_{(2,1,1,1,1,1,1,1)}$	–	–	$\frac{21678902901545427631303627 3}{67968750000000000000000000}$
6	$G_{(7);a,b}$	(3)	-1	$\frac{49317944388734216342783360 3}{13593750000000000000000000}$
7	$G_{(6,1);a,b}$	$(4,-1)$	-1	$\frac{81276582295211350007274538 9}{76464843750000000000000000}$
8	$G_{(5,2);a,b}$	$(0,3)$	-1	$\frac{62167895823868919398383790 542583}{70837031250000000000000000000000}$
9	$G_{(5,2);a,b}$	$(1,2)$	-1	$\frac{16349742672694910124756998 3295373}{23612343750000000000000000000000}$
10	$G_{(5,2);a,b}$	$(3,0)$	-1	$\frac{40898714423911742716816466 080817}{70837031250000000000000000000000}$
11	$G_{(5,1,1);a,b}$	$(0,1,-4)$	1	$\frac{31631429016033966500059901 3407}{24468750000000000000000000000000}$
12	$G_{(5,1,1);a,b}$	$(1,-4,0)$	1	$\frac{56165887401691366342797622 43}{30585937500000000000000000000000}$
13	$G_{(5,1,1);a,b}$	$(2,-2,3)$	-1	$\frac{11461831387776513756474092 1259}{24468750000000000000000000000000}$
14	$G_{(4,3);a,b}$	$(1,-1)$	0	$\frac{53391677322784020922289982 06713}{65589843750000000000000000000000}$
15	$G_{(4,2,1);a,b}$	$(1,-4,-3)$	2	$\frac{31073909255556627765856054 8518749}{70837031250000000000000000000000}$
16	$G_{(4,2,1);a,b}$	$(1,-3,-1)$	1	$\frac{39203457027911089552018539 0303}{34981250000000000000000000000000}$
17	$G_{(4,2,1);a,b}$	$(1,2,0)$	-1	$\frac{12299856119437862451844941 3573949}{14167406250000000000000000000000000}$
18	$G_{(4,2,1);a,b}$	$(3,-1,4)$	-2	$\frac{30884554274001200121651411 444769}{47224687500000000000000000000000}$
19	$G_{(4,2,1);a,b}$	$(4,-1,3)$	-2	$\frac{15321637693779276596484703 530539}{29515429687500000000000000000000}$
20	$G_{(3,3,1);a,b}$	$(0,3,0)$	-1	$\frac{19163177340430193809351035 52589}{98384765625000000000000000000}$
21	$G_{(3,3,1);a,b}$	$(1,-3,-1)$	1	$\frac{23562437161184671324402035 11597}{19676953125000000000000000000}$

r	H_r	a	b	w_r
22	$G_{(3,2,2);a,b}$	$(1,-3,-1)$	1	$\frac{4247733296085237013561021328468 9}{2098875000000000000000000000000}$
23	$G_{(3,2,2);a,b}$	$(1,1,4)$	-2	$\frac{123346068448463558503725606137}{678515625000000000000000000000}$
24	$G_{(3,2,2);a,b}$	$(3,-1,1)$	-1	$\frac{346574630387532893541592905802 69}{629662500000000000000000000000000}$
25	$G_{(3,2,1,1);a,b}$	$(0,1,-3,0)$	0	$\frac{111868341854332796257367494399}{135937500000000000000000000000}$
26	$G_{(2,2,2,1);a,b}$	$(1,0,0,-3)$	0	$\frac{129021577255464953947629819}{453125000000000000000000000}$
27	$G_{(2,1,1,1,1,1);a,b}$	$(1,-4,0,0,0,0)$	0	$\frac{223483833629053550920204781 47}{679687500000000000000000000000}$
28	$G_{(2,1,1,1,1,1);a,b}$	$(1,-3,0,0,0,0)$	0	$\frac{209379509697124009531312025 29}{339843750000000000000000000000}$

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