The multi-marginal converse of Minty's theorem fails in the plane

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Abstract

For the three-marginal quadratic cost on the real plane, we give three integer-coordinate tuples that are c-monotone but admit no compatible tuple whose coordinate sum is zero. Four necessary inequalities have a positive weighted sum bounded below by the exact constant $23/26$, as witnessed by a sum of four squares. Every maximal c-monotone extension of these tuples therefore has a proper, nonconvex sum image. This gives a negative answer to the implication asked about in Remark 2.6 of Bartz, Bauschke, Phan and Wang [1]: maximal c-monotonicity does not imply surjectivity of the sum map. We use their two-point definition of c-monotonicity. The finite obstruction, the cost-swap equivalence, the maximal-extension argument and the nonconvexity conclusion are formalized in Lean 4; the interpretation through firmly nonexpansive maps is proved only in the text.

Verification status. A fresh replay on 31 October 2026 compiled all six project modules and the audit module against a Mathlib checkout built from source at the pinned revision. All 47 declarations defined in the project modules, including 18 generated or auxiliary declarations, depend only on `propext`, `Classical.choice` and `Quot.sound`. The exact rational checker also passed; the record is in `verification/`.

1 The question and the precise conclusion

Let $H = \mathbb{R}^2$ with its ordinary Euclidean inner product, let $X = H^3$, and put

$$c_3(x_1, x_2, x_3) = \langle x_1, x_2\rangle + \langle x_1, x_3\rangle + \langle x_2, x_3\rangle, \qquad S(x) = x_1 + x_2 + x_3.$$

For $K \subseteq \{1, 2, 3\}$, let $m_K(x, y)$ take coordinate i from x when $i \in K$ and from y otherwise. A set $\Gamma \subset X$ is *c_3-monotone* if, for every $x, y \in \Gamma$ and every such K,

$$c_3(m_K(x, y)) + c_3(m_K(y, x)) \le c_3(x) + c_3(y). \tag{1}$$

This is two-point c-monotonicity. No claim about full cyclic c-monotonicity is made. Maximality means maximality by inclusion among all subsets of X satisfying (1).

Writing $d_i = x_i - y_i$, expansion of the four costs gives

$$c_3(x) + c_3(y) - c_3(m_K(x, y)) - c_3(m_K(y, x)) = \left\langle \sum_{i\in K} d_i, \sum_{i\notin K} d_i \right\rangle. \tag{2}$$

Consequently, for three marginals it is enough to check the three singleton cuts:

$$\langle x_i - y_i, S(x) - S(y) - (x_i - y_i)\rangle \ge 0, \qquad i = 1, 2, 3. \tag{3}$$

Complementary cuts give the same inner product. The empty and full cuts give zero. This agrees with Definition 1.2 and Lemma 2.3 of [1].

For a c_3-monotone set Γ, Bartz et al. prove that $S(\Gamma) = H$ implies maximal c-monotonicity, together with several equivalent full-domain resolvent conditions [1, Theorem 2.5]. Their Remark 2.6 asks whether maximality implies this sum-surjectivity. The following theorem negates that implication.

Theorem 1. *There is a three-element c_3-monotone set $\Gamma_0 \subset (\mathbb{R}^2)^3$ such that every c_3-monotone set $\Gamma \supseteq \Gamma_0$ satisfies*

$$0 \notin S(\Gamma), \qquad S(\Gamma) \text{ is not convex.}$$

There exists a maximally c_3-monotone such Γ. In particular, maximal c_3-monotonicity does not imply $S(\Gamma) = \mathbb{R}^2$.

The maximal set in this theorem is obtained by Zorn's lemma. The explicit object is the finite obstruction Γ_0; we do not assert that this three-element set is itself maximal.

2 The integer-coordinate obstruction

Define $\Gamma_0 = \{A, B, C\}$ by the following table.

Point	First marginal	Second marginal	Third marginal	Sum
A	$(-1, -2)$	$(3, -12)$	$(9, -2)$	$(11, -16)$
B	$(-2, -5)$	$(0, -7)$	$(-14, -2)$	$(-16, -14)$
C	$(2, 0)$	$(-4, 15)$	$(14, -1)$	$(12, 14)$

The three values on the left of (3) are as follows.

Pair	$i = 1$	$i = 2$	$i = 3$
A, B	11	57	92
A, C	50	25	9
B, C	211	4	27

Every entry is strictly positive. Reversing a pair changes both factors in the inner product by a sign, and equal pairs give zero. Thus the table and (2) prove c_3-monotonicity for every pair and every cut.

Lemma 2. *No $U = (u, v, w) \in (\mathbb{R}^2)^3$ with $u + v + w = 0$ is c_3-monotonically related to all three points of Γ_0.*

Proof. Write $u = (a, b)$, $v = (c, d)$ and $w = (e, f)$. Fix a seed point P with sum $s = S(P)$ and one marginal i. Since $S(U) = 0$, the corresponding inequality (3) is equivalent to

$$\|U_i - P_i\|^2 + \langle U_i - P_i, s\rangle \leq 0. \tag{4}$$

Using $(P, i) = (A, 1), (B, 2), (B, 3), (C, 2)$ respectively, expansion gives the four necessary inequalities

$$\begin{aligned}
q_{A1} &= a^2 + b^2 + 13a - 12b - 16 \leq 0,\\
q_{B2} &= c^2 + d^2 - 16c - 49 \leq 0,\\
q_{B3} &= e^2 + f^2 + 12e - 10f - 52 \leq 0,\\
q_{C2} &= c^2 + d^2 + 20c - 16d + 79 \leq 0.
\end{aligned}$$

The zero-sum hypothesis gives $e = -a - c$ and $f = -b - d$. Direct polynomial expansion then establishes the identity

$$4q_{A1} + 3q_{B2} + 4q_{B3} + 8q_{C2} = 8\left(a + \frac{c}{2} + \frac{1}{4}\right)^2 + 8\left(b + \frac{d}{2} - \frac{1}{2}\right)^2 + 13\left(c + \frac{31}{13}\right)^2 + 13\left(d - \frac{42}{13}\right)^2 + \frac{23}{26}. \tag{5}$$

The left side is nonpositive, since its four weights are positive. The right side is at least $23/26 > 0$. This contradiction proves the lemma. No numerical optimizer or numerical tolerance is used in this argument. □

3 Maximality and a nonconvex sum image

Proof of Theorem 1. The preceding table proves that Γ_0 is c_3-monotone. If a c_3-monotone superset Γ contained a tuple of sum zero, that tuple would be compatible with all of Γ_0, contradicting Lemma 2. Thus $0 \notin S(\Gamma)$ for every such superset.

Consider the family of all c_3-monotone supersets of Γ_0, ordered by inclusion. It is nonempty. The union of a nonempty chain belongs to the family: any two tuples in the union lie together in one chain member, where every inequality (1) holds. Zorn's lemma supplies a maximal family member Γ. Any c_3-monotone proper superset of Γ would still contain Γ_0, contradicting this maximality. Hence Γ is maximal among all c_3-monotone sets.

Finally, the three sums satisfy the exact relation

$$28S(A) + 173S(B) + 205S(C) = 0, \qquad 28 + 173 + 205 = 406.$$

All three weights are positive. If $S(\Gamma)$ were convex, it would contain their normalized convex combination zero. This contradicts the already established exclusion. The argument applies to every c_3-monotone superset, including the maximal one. □

4 The simultaneous-extension obstruction

The relation to Minty parametrization is explicit. If x, y belong to a c_3-monotone set and $S(x) = S(y)$, inequality (3) reads $-\|x_i - y_i\|^2 \geq 0$ for each i, hence $x = y$. Thus on $D = S(\Gamma)$ there are well-defined component maps $J_i(S(x)) = x_i$. They satisfy

$$\sum_{i=1}^{3} J_i(s) = s, \qquad \|J_i(s) - J_i(t)\|^2 \leq \langle J_i(s) - J_i(t), s - t\rangle.$$

These are precisely the firmly nonexpansive inequalities. For the three-point domain $D_0 = \{S(A), S(B), S(C)\}$, the displayed data therefore give a partition of the identity into three firmly nonexpansive maps. There is no simultaneous extension of these maps to $D_0 \cup \{0\}$ that preserves both firm nonexpansiveness and the identity-sum condition. Indeed, the three values at zero would form the tuple excluded by Lemma 2. Equivalently, the maps $Q_i = 2J_i - \mathrm{Id}$ are nonexpansive on D_0 and sum to $-\mathrm{Id}$, but cannot be extended jointly to zero while retaining these properties.

This consequence concerns compatibility of the three extensions. It does not assert a failure of the classical extension or Minty theorem for one ordinary monotone operator.

5 Prior work and historical limits

The target is the implication in [1, Remark 2.6], not the different question about contact sets of mutually c-conjugate potentials. Lim's multidimensional example [2, Proposition 1.10 and

Example 4.1] has a hexagonal gap in its sum image, but Lim explicitly shows in Step 5 that its contact set is not maximally c-monotone and can be enlarged by the zero tuple. In the present construction, every c_3-monotone extension is prevented from adding any zero-sum tuple. The distinction is essential to answering the maximality question.

The 2025 survey [3] discusses the multimarginal setting and cites [1]; it does not supply the finite obstruction above. Finite incompatibility certificates, constrained Lipschitz extension obstructions, and sum-of-squares identities are established techniques. For example, Dacorogna and Gangbo [4] study finite obstructions for bounded-strain extension problems, whose hypotheses differ from the simultaneous firmly nonexpansive identity-partition condition here. No novelty is claimed for those general mechanisms. The simplex/difference-body extremizers of classical Jung geometry also give the hexagonal radius ratio 2/3; see [5, Lemma 2.9 and Corollary 2.10]. A separate three-dimensional obstruction considered during this work is a short lift of that known geometry. Its existence is not evidence of a new extension method. Whether earlier constrained-section results already yield the present maximality conclusion remains a historical question.

Searches in October 2026 of the arXiv, citing records and code repositories found no earlier answer to the question of Remark 2.6; Bartz listed a multi-marginal Minty characterization among open questions in 2020 and 2021 conference talks. The search is bounded, no external expert has assessed the result, and historical priority is not certified.

6 Formal verification

The formal development uses Lean 4.19.0 and the following mathlib commit:

`c44e0c8ee63ca166450922a373c7409c5d26b00b`.

It defines the real coordinate space, the Euclidean pairing, the actual cost-swap predicate and inclusion-maximality. It verifies the table, the polynomial identity, exclusion of every real zero-sum tuple, equivalence with all cost swaps, the chain-union and Zorn argument, and the actual nonconvexity of the sum image. The principal endpoint is

`Minty.exists_maximal_cMonotone_nonconvex_sum`.

The module `Minty.AxiomAudit` enumerates every declaration whose defining module is one of the six project modules, including private and generated declarations, and rejects any axiom other than `propext`, `Classical.choice` and `Quot.sound`. The script `scripts/check_axiom_audit.py` checks that output, and `lean/CORRESPONDENCE.md` maps each step above to its declarations. The standard-library script `certificate/verify_minty_exact.py` checks the table and the polynomial identity in exact arithmetic. The interpretation through firmly nonexpansive maps in the previous section is proved in the text and is not packaged separately in Lean.

7 Limitations

The theorem concerns three marginals on the real plane and the two-point notion of c-monotonicity of [1]. It does not classify other numbers of marginals or dimensions, assert minimality of the dimension, or address full cyclic c-monotonicity or c-splitting potentials. The maximal set is obtained by Zorn's lemma and is not described explicitly. Lean verification establishes only the formal statements described in Section 6, and the literature comparison does not establish historical priority.

8 Research process and AI disclosure

This is fully autonomous agentic research. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed, no expert validation is claimed, and the novelty assessment is provisional.

The question selection, the counterexample, the proof, the Lean formalization, the literature searches and an earlier version of this note were produced by autonomous AI research agents in an earlier session. The records of that session state that it used an autonomous OpenAI GPT-6 Astra workflow with delegated discovery and review roles, and that one additional review used GPT-6.1 Sol. A numerical search suggested the witness; no solver output enters the proof.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It read Remark 2.6 and Definition 1.2 of [1] and the relevant parts of [2], compared the Lean definitions of the cost, the swap condition and maximality with them, rechecked the table, the identity and the convex combination in exact arithmetic with its own code, ran the fresh Lean replay, and revised this note, including the editorial revision of 1 November 2026, which did not change the mathematics. AI checks are not human peer review. Section 6 states the formal scope and Section 7 the remaining limits.

References

[1] S. Bartz, H. H. Bauschke, H. M. Phan and X. Wang, *Multi-marginal maximal monotonicity and convex analysis*, Mathematical Programming **185** (2021), 385–408; published online in 2019. doi:10.1007/s10107-019-01433-9. Preprint arXiv:1901.03777v2.

[2] T. Lim, *Maximal monotonicity and cyclic involutivity of multiconjugate convex functions*, SIAM Journal on Optimization **33** (2023), 2489–2511. doi:10.1137/23M1549201. Preprint arXiv:2207.04830v4.

[3] L. De Pascale, A. Kausamo and K. Wyczesany, *60 years of cyclic monotonicity: a survey*, Journal of Convex Analysis **32** (2025), 399–430. doi:10.68381/jca32020. Preprint arXiv:2308.07682.

[4] B. Dacorogna and W. Gangbo, *Extension theorems for vector valued maps*, Journal de Mathématiques Pures et Appliquées **85** (2006), 313–344. doi:10.1016/j.matpur.2005.04.005.

[5] R. Brandenberg and B. González Merino, *The asymmetry of complete and constant width bodies in general normed spaces and the Jung constant*, 2015 preprint version. arXiv:1412.8693v3.