The exact asymmetric Merino–Welsh region for finite matroids

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Abstract

For which nonnegative real pairs (x, y) does $T_M(x,0)T_M(0,y) \geq T_M(1,1)^2$ hold for every finite matroid without loops or coloops? Liu determined the sharp threshold on the diagonal $x = y$ [1], and Chen and Xie gave a direct inductive proof [2]. We show that the answer is exactly $x > 1$, $y > 1$, and

$$xy^k \geq \left(\sqrt{x-1} + k\sqrt{y-1}\right)^2, \qquad yx^k \geq \left(\sqrt{y-1} + k\sqrt{x-1}\right)^2$$

for every integer $k \geq 2$. Parallel thickenings of uniform matroids give necessity. For sufficiency, the scalar conditions support a parallel-class induction, including the case where deletion creates coloops. The characterization includes parameters with one coordinate below two. A Lean formalization proves the equivalence for actual finite Mathlib matroids, with the Tutte polynomial given by the literal subset rank sum.

1 The parameter question

The Tutte polynomial records rank information about all subsets of a matroid. Let M be a finite matroid with ground set E and rank function r. Use

$$T_M(u,v) = \sum_{S \subseteq E} (u-1)^{r(E)-r(S)}(v-1)^{|S|-r(S)}, \qquad b(M) = T_M(1,1).$$

The number $b(M)$ counts the bases of M. Call M *admissible* if it has no loops and no coloops. The empty matroid is admissible and has $T_M = 1$. Write $\mathcal{U}(x,y)$ for the assertion

$$T_M(x,0)T_M(0,y) \geq b(M)^2 \quad \text{for every finite admissible } M. \tag{1}$$

No representability or connectedness condition is imposed.

The diagonal question asks when $\mathcal{U}(c,c)$ holds. Liu proved its sharp threshold [1]. Chen and Xie gave a direct inductive proof [2]. The threshold is the largest real root of $c^3 = 9(c-1)$. The obstruction comes from parallel thickenings of uniform matroids [3]. Here the two axis parameters can differ.

Theorem 1. *Let $x, y \geq 0$. Then $\mathcal{U}(x,y)$ holds if and only if $x > 1$, $y > 1$, and, for every integer $k \geq 2$,*

$$xy^k \geq \left(\sqrt{x-1} + k\sqrt{y-1}\right)^2, \tag{2}$$

$$yx^k \geq \left(\sqrt{y-1} + k\sqrt{x-1}\right)^2. \tag{3}$$

These scalar conditions are denoted by $\mathcal{R}(x,y)$. All square roots are nonnegative. Equality in either scalar condition is allowed. Thus the universal statement over matroids has an explicit description using only real parameters and positive integer indices.

The structural induction follows the approach of Chen and Xie. The axis comparison and the uniform-thickening obstruction are prior work. The result here is the full asymmetric characterization. Its sufficiency requires additional scalar estimates, especially when one coordinate lies between one and two. Section 6 gives a valid point $(3/2, 8)$ and shows why the two conditions at $k = 2$ alone do not suffice.

2 Rank identities and the axis comparison

The proof uses deletion–contraction, duality, and multiplication on direct sums. These identities follow from the rank sum by partitioning subsets or passing to complements. If e is neither a loop nor a coloop, then

$$T_M = T_{M\setminus e} + T_{M/e}.$$

A loop contributes a factor v, and a coloop contributes a factor u. Induction therefore gives nonnegativity and coordinatewise monotonicity for nonnegative arguments. Duality gives $T_{M^*}(u,v) = T_M(v,u)$. At $(1,1)$ only spanning independent subsets contribute. In particular, $b(M)$ is positive, including for the empty matroid.

The following comparison is the structural estimate used in the sharp diagonal proofs [1, 2].

Lemma 2. *If e is neither a loop nor a coloop and $z \geq 0$, then*

$$zT_{M/e}(z+1,0) \leq T_{M\setminus e}(z+1,0).$$

Proof. For any $e \in E$, define

$$F_M^e(z) = \sum_{\substack{S\subseteq E\setminus\{e\} \\ r(S\cup\{e\})=r(S)}} z^{r(E)-r(S)}(-1)^{|S|-r(S)}.$$

This sum is nonnegative. To prove this, induct on $|E\setminus\{e\}|$. On a singleton ground set it is either zero or one. For another element f, partition the sum according to membership of f. A loop makes the two parts cancel. A coloop gives $(z+1)F^e_{M\setminus f}(z)$. An ordinary element gives $F^e_{M\setminus f}(z) + F^e_{M/f}(z)$. The rank changes in deletion and contraction give each recurrence term by term. The induction proves nonnegativity in every case.

Now suppose e is ordinary. Partition subsets of $E\setminus\{e\}$ according to whether they span e. The nonspanning contributions to $T_{M\setminus e}(z+1,0) - zT_{M/e}(z+1,0)$ cancel. For the spanning subsets, contraction lowers rank by one and reverses the sign of the power of -1. Hence the difference is $(z+1)F^e_M(z) \geq 0$. □

Apply the same lemma to the dual. With $D = M\setminus e$, $Q = M/e$, and $x, y > 1$, the two comparisons are

$$(x-1)T_Q(x,0) \leq T_D(x,0), \qquad (y-1)T_D(0,y) \leq T_Q(0,y). \tag{4}$$

3 Necessity of the scalar region

3.1 An upper bound and parallel classes

If a matroid N has m elements and rank r, then, for $z > 0$,

$$T_N(z+1,0)z^{m-r} \leq (z+1)^m. \tag{5}$$

Induct on m. The empty case is equality. A loop makes the left side zero. A coloop reduces m and r by one and contributes $z+1$. For an ordinary element, the two minors have sizes $m-1$ and ranks $r, r-1$. Their contributions are bounded by $z(z+1)^{m-1}$ and $(z+1)^{m-1}$, which sum to the right side.

Suppose elements have q distinct labels, with elements sharing a label parallel. Deleting a duplicate preserves rank and the axis evaluation. Indeed, contracting it creates a loop and thus contributes zero at the second argument zero. Repeated deletion leaves q elements. Therefore

$$T_N(z+1,0)z^{q-r} \leq (z+1)^q. \tag{6}$$

3.2 Uniform thickenings

Let $P = P(r, n, k)$ have ground set $[n] \times [k]$ and rank $r_P(S) = \min\{r, |\pi(S)|\}$, where $\pi(i, j) = i$. This is the k-fold parallel thickening of $U_{r,n}$. For $1 \le r \le n$ and $k \ge 2$, it is admissible. Each element is a nonloop and has a distinct parallel copy, so deletion preserves rank. Choosing r labels and one copy per label gives

$$b(P) \ge \binom{n}{r} k^r. \tag{7}$$

The injection that makes this choice is enough; no asymptotic estimate for basis counts is needed.

Suppose $\mathcal{U}(x, y)$, with $x, y > 1$, and put $X = x - 1$, $Y = y - 1$. Apply (6) to P and (5) to P^*. Duality gives

$$T_P(x, 0)X^{n-r} \le x^n, \qquad T_P(0, y)Y^r \le y^{nk}.$$

Using (1) and (7), then taking square roots, gives

$$\binom{n}{r} a^{n-r} c^r \le s^n, \qquad a = \sqrt{X}, \quad c = k\sqrt{Y}, \quad s = \sqrt{xy^k} > 0.$$

The $r = 0$ term has the same bound because $X \le xy^k$. The binomial theorem now yields $(a + c)^n \le (n + 1)s^n$ for every n. If $(a + c)/s > 1$, its powers eventually exceed $n + 1$, a contradiction. Thus $a + c \le s$, which proves (2). Duality implies $\mathcal{U}(y, x)$ and proves (3).

3.3 Coordinates at most one are impossible

Suppose $\mathcal{U}(x, y)$ with $x, y \ge 0$ and $x \le 1$. Choose an integer $n > y+2$ and the rank-one matroid P on n parallel elements. Its basis count is at least n. Apply the universal inequality to P^*. Monotonicity gives $T_P(0, x) \le T_P(1, 1) = b(P)$. Equation (6), with one label and rank one, gives $T_P(y + 2, 0) \le y + 2$. Consequently,

$$b(P)^2 \le T_P(0, x)T_P(y, 0) \le b(P)T_P(y + 2, 0) \le b(P)(y + 2),$$

contrary to $b(P) > y+2$. Hence $x > 1$. Interchanging the coordinates gives $y > 1$ and completes necessity.

4 Scalar estimates for sufficiency

Assume $\mathcal{R}(x, y)$. Throughout this section, set

$$X = x - 1 > 0, \quad Y = y - 1 > 0, \quad a = \sqrt{X}, \quad b = \sqrt{Y}, \quad S_k(y) = \sum_{j=0}^{k-1} y^j,$$

and $H_k = (1+a^2)(1+b^2)^k - (a+kb)^2$. The hypotheses state $H_k \ge 0$ and its coordinate-swapped counterpart for $k \ge 2$.

4.1 The shifted product exceeds one

Lemma 3. *$ab > 1$. In particular, $XY > 1$ and $xy > 4$.*

Proof. The forward inequalities also hold for $k = 0, 1$, since $H_0 = 1$ and $H_1 = (ab - 1)^2$. Put $B = b^2$ and $q = (1 + 2B)^{-1}$. Both q and $(1 + B)q$ lie in $(0, 1)$. Multiply $H_k \ge 0$ by q^k and sum. The geometric moment identities give

$$\frac{a^2}{1 - q} + \frac{2abq}{(1 - q)^2} + \frac{Bq(1 + q)}{(1 - q)^3} \le \frac{1 + a^2}{1 - (1 + B)q}.$$

The left side minus the right side equals $\frac{1+2B}{4B^2}(1+2ab-2a^2b^2-3B)$. Thus $2a^2b^2+3B \geq 2ab+1$.

Suppose $ab \leq 1$. By symmetry take $b \leq 1$ and write $t = ab \in (0,1]$. Since $t^2 \leq t$, the last inequality implies $B \geq 1/3$. If $b = 1$, the condition $H_2 \geq 0$ gives $3a^2 - 4a \geq 0$, impossible for $0 < a \leq 1$. Hence $1/3 \leq B < 1$. Multiplying $H_3 \geq 0$ by B gives $F(t) \geq 0$, where

$$F(t) = (t^2 + B)(1 + B)^3 - (t + 3B)^2.$$

This quadratic is convex in t. On $[0,1]$ it is at most $(1-t)F(0) + tF(1)$. But

$$F(1) = B(B-1)(B^2+5B+2) < 0, \qquad F(0) = B\big((1+B)^3 - 9B\big) < 0.$$

The second sign follows from the exact identity

$$(1+B)^3 - 9B = (B-1)B^2 + 4(B-1/3)(B-1) - \tfrac{2}{3}B - \tfrac{1}{3}.$$

Every term is nonpositive and the last is negative. This contradiction proves $ab > 1$. Finally $xy = 1 + X + Y + XY > 4$, since $X + Y \geq 2\sqrt{XY} > 2$. □

4.2 The parallel-class estimate

Lemma 4. *Let $k \geq 2$ and $S = S_k(y)$. Suppose $A, B_0, U, V, s, t \geq 0$ satisfy*

$$A \geq XB_0, \qquad V \geq YU, \qquad AU \geq s^2, \qquad B_0V \geq t^2.$$

Then $(A + B_0)(U + SV) \geq (s + kt)^2$.

Proof. First let $B_0, V > 0$. Put $p = \sqrt{A/B_0}$ and $q = \sqrt{U/V}$. Then $p \geq a$, $0 \leq bq \leq 1$, $ab > 1$, and $S \geq k$. The difference $(A + B_0)(U + SV) - (\sqrt{AU} + k\sqrt{B_0V})^2$, divided by B_0V, is $F = q^2 + Sp^2 + S - k^2 - 2kpq$. Using $b^2S = y^k - 1$ gives the identity

$$b^2F = H_k + (bq - 1)(bq + 1 - 2kbp) + b(p - a)\big(Sb(p + a) - 2k\big). \tag{8}$$

Both factors in the middle product are nonpositive, since $bp > 1$. The final product is nonnegative because $b(p + a) \geq 2ab > 2$ and $S \geq k$. Thus $F \geq 0$. The hypotheses give $\sqrt{AU} \geq s$ and $\sqrt{B_0V} \geq t$, proving the result. If $B_0 = 0$, then $t = 0$ and the result follows from $AU \geq s^2$. If $V = 0$, then $U = 0$ and $s = t = 0$. These cases include all zeros. □

4.3 The estimate for new coloops

Lemma 5. *For every $k \geq 2$,*

$$xS_k(y) > (k + 1/2)^2, \qquad x(S_k(y) - 1) \geq k^2.$$

If $A, B_0, V, s, t \geq 0$, $A \geq XB_0$, $B_0V \geq t^2$, and $2s \leq t$, then $(A + B_0)S_k(y)V \geq (s + kt)^2$.

Proof. Write $\tau = \sqrt{XY} > 1$. The swapped $k = 2$ condition gives

$$(X + 2)\tau^2 - 4\tau + (X - 1)^2 \geq 0. \tag{9}$$

Suppose $x(1 + y) \leq 25/4$. Then $4(X + 1)\tau^2 \leq X(17 - 8X)$. Multiply (9) by $4(X + 1)$ and use this bound and $\tau > 1$. The result is

$$\begin{aligned} 0 &< X(X + 2)(17 - 8X) - 16(X + 1) + 4(X + 1)(X - 1)^2 \\ &= -\big(4X^3 + 3X^2 - 14X + 12\big) < 0. \end{aligned}$$

The last sign follows from $4X^3 + 3X^2 - 14X + 12 = (X-1)^2(4X+11) + 4X + 1 > 0$. This contradiction proves the first bound at $k = 2$. For $k \geq 2$, (2) gives

$$xy^k \geq (a + kb)^2 \geq 4kab > 4k. \tag{10}$$

Since $S_{k+1} = S_k + y^k$ and $(k + 3/2)^2 - (k + 1/2)^2 = 2k + 2 \leq 4k$, induction proves the first bound. The second starts from $xy > 4$ at $k = 2$ and uses $(k+1)^2 - k^2 = 2k + 1 \leq 4k$ with the same increment. Finally,

$$(A + B_0)S_k(y)V \geq xB_0S_k(y)V \geq xS_k(y)t^2 \geq (k + 1/2)^2t^2 \geq (s + kt)^2.$$

□

A loopless rank-one matroid on k elements has $T_M(u, v) = u - 1 + S_k(v)$. If it is admissible then $k \geq 2$. The second bound in Lemma 5 proves (1) for this matroid.

5 Sufficiency by matroid induction

The proof of (1) for both (x, y) and (y, x) uses strong induction on the number of elements, assuming $\mathcal{R}(x, y)$. The empty case is equality. All reductions below use ranks and therefore do not require connectedness or representability.

5.1 An element without a parallel or series mate

Suppose an element e belongs to no two-element circuit or cocircuit. Its deletion D and contraction Q are admissible. Contraction can create a loop only from a parallel mate; deletion can create a coloop only from a series mate. Both minors have fewer elements, so induction applies. For nonnegative A, B_0, U, V,

$$(\sqrt{AU} + \sqrt{B_0V})^2 \leq (A + B_0)(U + V),$$

because the difference is $(\sqrt{AV} - \sqrt{B_0U})^2$. Use the two axis evaluations of D, Q in this inequality. Induction bounds the two square roots by their basis counts. Deletion–contraction then proves the required inequality for M.

5.2 Collapse of a maximal parallel class

It remains to consider the case where every element has a parallel or series mate. Choose a pair, dualizing and swapping x, y if needed to make it parallel. Let P be its maximal parallel class, $k = |P| \geq 2$, and choose $e \in P$. Put

$$C = E \setminus P, \qquad L = M|C, \qquad N = M|(C \cup \{e\}).$$

The class P has rank one. Submodularity shows that adding its elements to a set that contains e cannot raise rank. Thus

$$r(S \cup P) = r(S \cup \{e\}) \quad (S \subseteq E). \tag{11}$$

In particular $r(N) = r(M)$. Applying the same identity after removing $f \in C$ shows that N has no possible coloop other than e.

If $r(C) < r(M)$, then $r(P) + r(C) = r(M)$. This makes P a separator, so M is the direct sum of its restrictions to P and C. For a direct rank proof, let $A \subseteq P$, $B \subseteq C$. Submodularity gives

$$r(P \cup B) + r(A) \leq r(P) + r(A \cup B), \quad r(E) + r(B) \leq r(C) + r(P \cup B).$$

Adding proves $r(A) + r(B) \leq r(A \cup B)$. The reverse inequality follows from submodularity on disjoint sets. Hence the rank sum factors. Both factors are admissible, since a coloop of either would be a coloop of M. The rank-one case and induction prove the product inequality.

Now suppose $r(C) = r(M)$. The matroid N is admissible, as is $Q = N/e$. Maximality of P excludes loops in Q, and contraction creates no coloops. The matroids L, Q have fewer elements than M. The matroid L is loopless but can have coloops. Removing the parallel copies successively gives

$$T_M(u, v) = T_L(u, v) + S_k(v)T_Q(u, v). \tag{12}$$

Indeed, contracting one of j copies creates $j - 1$ loops and leaves Q, so its contribution is $v^{j-1}T_Q$. Summing these contributions proves (12).

Set $A = T_L(x, 0)$, $B_0 = T_Q(x, 0)$, $U = T_L(0, y)$, $V = T_Q(0, y)$, $s = b(L)$, and $t = b(Q)$. Equation (12) gives

$$T_M(x, 0) = A + B_0, \quad T_M(0, y) = U + S_k(y)V, \quad b(M) = s + kt. \tag{13}$$

The axis comparisons in N give $A \geq XB_0$ and $V \geq YU$. If L has no coloops, induction gives $AU \geq s^2$ and $B_0V \geq t^2$. Lemma 4 and (13) prove the result.

5.3 When deletion creates a coloop

Suppose f is a coloop of L. It has no parallel mate in M. A mate in P would put f in P, and a mate in C would contradict its coloop status in L. Therefore f has a series mate g. This mate is outside P. To see this, write $R = r(M)$. The series pair gives $r(E \setminus \{f, g\}) = R - 1$, whereas $r(E \setminus \{f\}) = R$. If $g \in P$, another element of P remains in the first set. Adding its parallel mate g cannot raise rank, contradicting those two equalities.

The element g is also a coloop of L. Put $D = C \setminus \{f, g\}$. Since f is a coloop, $r(C \setminus \{f\}) = R-1$ and $r(C \setminus \{g\}) = r(D) + 1$. Apply submodularity to $E \setminus \{f, g\}$ and $C \setminus \{f\}$. Their union is $E \setminus \{f\}$ and intersection is D, so $R + r(D) \leq 2(R - 1)$. Hence $r(C \setminus \{g\}) \leq R - 1$. Deleting one element lowers rank by at most one, giving equality.

The required bound is $b(Q) \geq 2b(L)$. Every basis B of L contains f, g. For $h = f$ or $h = g$, the set $B \setminus \{h\}$ is a basis of Q. It has size and rank $R - 1$ in L. To see that e raises its rank, apply submodularity to $(B \setminus \{h\}) \cup \{e\}$ and $C \setminus \{h\}$. Their union is $E(N) \setminus \{h\}$ of rank R, and their intersection has rank $R - 1$. Since $C \setminus \{h\}$ has rank $R - 1$, the first set has rank R. Contraction by e gives the claim. Each deletion map is injective because reinserting h recovers B. The two images are disjoint, since exactly the image obtained by deleting f omits f. Thus $2s \leq t$.

A coloop makes $U = 0$. Induction for Q gives $B_0V \geq t^2$. Lemma 5 and (13) prove the result. This completes sufficiency and Theorem 1.

6 Examples and scope

At $(x, y) = (301, 101/100)$, both coordinates exceed one and $XY = 3 > 1$. Both $k = 2$ constraints hold, but the forward $k = 3$ constraint fails. For exact bounds, use $433/25 < \sqrt{300} < 35/2$. The forward $k = 2$ difference is

$$\frac{70101}{10000} - \frac{2}{5}\sqrt{300} > \frac{101}{10000} > 0.$$

For the swapped condition, the product is 91507.01, while the square is at most 1232.01. At $k = 3$, the square exceeds $17.62^2 = 310.4644$, but $xy^3 = 310.120601$. These terminating decimals denote exact rational numbers. Thus the conditions for larger k carry information that the two $k = 2$ conditions do not contain.

The known point $(3,3)$ [4] satisfies the region conditions because $2(k+1)^2 \le 3^{k+1}$ for $k \ge 2$. Check $k=2$ and use $3(k+1)^2 \ge (k+2)^2$ for induction. The point $(3/2, 8)$ also belongs to the region. With $a = \sqrt{1/2}$ and $b = \sqrt{7}$,

$$(a+kb)^2 \le 1 + 14k^2 \le \tfrac{3}{2}8^k, \qquad (b+ka)^2 \le 14 + k^2 \le 8(3/2)^k.$$

The power bounds hold at $k=2$. Their induction steps follow from

$$8(1+14k^2) - (1+14(k+1)^2) = 98k^2 - 28k - 7 \ge 0$$

and

$$\tfrac{3}{2}(14+k^2) - (14+(k+1)^2) = \tfrac{1}{2}(k-2)^2 + 4 > 0.$$

The square estimates follow from $(u-v)^2 \ge 0$. These examples show nonvacuity and the range below two. They do not claim a sharp boundary point.

The theorem covers the nonnegative quadrant. It does not classify negative parameters or every equality case. It also does not identify which integer indices are active on the boundary, or assert that every index is necessary. The graph Merino–Welsh problem at $(2,2)$ is unchanged.

7 Formal statement and reproduction

The main declaration is `AsymmetricTutte.universal_iff_region`. It uses actual Mathlib matroids with finite ground sets. The ground set can be a proper subset of an infinite ambient type. The Tutte polynomial is the rank-generating subset sum given above. The intermediate structure `ScalarInputs` collects the induction estimates. The region assumptions prove every field of that structure, so it is not an extra hypothesis in the final equivalence.

The source package pins Lean 4.32.1, Mathlib, and Liu's proof library. Its README gives reproduction commands and the exact dependency revisions. The external proof library is fetched from its public repository and is not redistributed. The final axiom audit permits only the standard foundation constants `propext`, `Classical.choice`, and `Quot.sound`.

8 Research process and AI disclosure

This work is fully autonomous agentic research. As recorded for this result, the initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT, and GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities; this check is not expert validation or peer review. The editorial revision of 1 November 2026 was made with Opus 5.5 by Anthropic used with Claude Code; it changed the presentation only, not the mathematics or the formalization.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed. Section 7 states the formal scope, and Section 6 the limits of the theorem.

References

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