A five-variable separation between structural and numerical selfadhesivity

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30 October 2026; revised 1 November 2026

Abstract

We give an explicit five-variable conditional-independence model that admits a structural adhesive extension at every overlap, yet is not induced by any numerically selfadhesive real polymatroid. The positive statement is certified by 32 integer polymatroids, with all nontrivial coordinate tables printed below. The obstruction is an exact identity involving 51 nonnegative conditional mutual informations and two substantive copied extensions. It excludes every real inducing profile, not merely the profiles used in the positive certificates. This gives a negative answer to Question 4.18 (4.14 in the PDF version) of Boege's 2023 paper on selfadhesivity [1]. The complete theorem, its powerset semantics, and the exact certificates are formalized in Lean 4 for arbitrary real polymatroids. Formal correctness and historical novelty are separate claims, and the novelty assessment is provisional.

Verification status. A fresh replay on 30 October 2026 compiled all 47 modules one at a time, with one Lean thread, from a copy of the sources containing no compiled project objects, against the pinned dependencies with Mathlib objects from the official cache. The 28 audited endpoints, including both main theorems, depend only on `propext`, `Classical.choice` and `Quot.sound`. The exact certificate checkers were rerun and passed; the record is in `verification/`.

Novelty. No earlier answer to the question was found in the literature searched, but coverage is bounded and no specialist has assessed the result. No historical priority is asserted. An earlier counterexample in another AI research repository resolves a neighbouring selfadhesivity question and is discussed in Section 4.

1 The exact question and theorem

Let $N = \{1, 2, 3, 4, 5\}$. A polymatroid is a function $h : 2^N \to \mathbb{R}$ satisfying

$$h(\varnothing) = 0, \qquad h(A) \le h(B) \ (A \subseteq B), \qquad h(A) + h(B) \ge h(A \cap B) + h(A \cup B).$$

For distinct i, j and $K \subseteq N \setminus \{i, j\}$, put

$$I_h(i; j \mid K) = h(Ki) + h(Kj) - h(K) - h(Kij).$$

Juxtaposition denotes union. Every such quantity is nonnegative. The induced CI model $\mathcal{M}_h$ is its exact zero set. Thus specifying $\mathcal{M}_h = \mathcal{M}$ requires strict positivity at every elementary CI outside $\mathcal{M}$.

For $L \subseteq N$, an L-copy is a set N'_L together with a bijection $b_L : N \to N'_L$ that fixes L, where $N \cap N'_L = L$. A polymatroid g on $U_L = N \cup N'_L$ is adhesive at this overlap when

$$A_L(g) := g(N) + g(N'_L) - g(U_L) - g(L) = 0. \tag{1}$$

This is the group conditional independence of the two complements given L. Adhesivity of polymatroids in this sense was studied by Matúš [3]. The implication from adhesion to all required elementary crossing CIs is proved below.

A CI model is *structurally selfadhesive* if, for every L, such a g_L exists with both marginal CI models exactly equal to $\mathcal{M}$. Its inducing numerical profile can depend on L. A single polymatroid h is *numerically selfadhesive* if for every L an adhesive g_L exists whose two numerical restrictions are both exactly h under the copy map.

Boege asks whether every structurally selfadhesive structural CI model has a numerically selfadhesive inducing polymatroid [1, Question 4.18 in the HTML; Question 4.14 in the PDF]. These are different quantifier orders:

$$\forall L\ \exists g_L \qquad (\mathcal{M}_{g_L|_N} = \mathcal{M}_{g_L|_{N'_L}} = \mathcal{M}),$$
$$\exists h\ \forall L\ \exists g_L \quad (\mathcal{M}_h = \mathcal{M}, \quad g_L|_N = g_L|_{N'_L} = h).$$

The copied restrictions are identified with profiles on N via b_L; polymatroid and adhesion conditions are understood in both lines. The theorem disproves their equivalence.

Theorem 1. *Let $\mathcal{M}$ consist exactly of the 26 CI statements in Table 1. Then:*

1. *For every $L \subseteq N$ and every bijective L-copy, there is a real polymatroid g_L on their union satisfying (1), whose two restrictions induce exactly $\mathcal{M}$.*
2. *There is no real polymatroid $h : 2^N \to \mathbb{R}$ inducing exactly $\mathcal{M}$ that is numerically selfadhesive at every overlap.*

Consequently the known inclusion of CI models induced by numerically selfadhesive polymatroids into $\mathfrak{st}^\diamond(N)$ is strict when $|N| = 5$.

Pair i, j	Conditioning sets K for which $I_h(i; j \mid K) = 0$
$1, 2$	$\varnothing$; 45
$1, 3$	4; 5; 24; 25; 45
$1, 4$	3; 23; 25
$1, 5$	3; 23; 24
$2, 3$	$\varnothing$; 45
$2, 4$	$\varnothing$
$2, 5$	$\varnothing$; 134
$3, 4$	12; 25
$3, 5$	12; 24
$4, 5$	$\varnothing$; 3; 12; 23

Table 1: The complete CI model. All remaining 54 elementary CMIs are strictly positive in an inducing polymatroid.

The encoding in the format of CInet [6] is

`01111101110000011101001111010011011111010111111101111110111101011111010101100101.`

Here 0 means CI. Pairs are ordered lexicographically; conditioning subsets of the other three variables are ordered first by size, then lexicographically. The encoding is auxiliary: the literal table fixes the theorem independently.

2 Positive proof: all overlaps have exact witnesses

We use canonical labels $U_L = \{1, \dots, 10 - |L|\}$. The original labels are $1, \dots, 5$. The copy fixes the labels of L and sends the remaining labels, in increasing order, to $6, 7, \dots, 10 - |L|$. Encode a subset S by $\sum_{i \in S} 2^{i-1}$. Every array below is indexed in increasing mask order, starting at the empty set.

We specify integer standardized supermodular tables m_L and set

$$g_L(S) = 1155\,|S| - m_L(S). \tag{2}$$

The modular term does not change any elementary CMI. The data are complete: Appendix B gives the 20 tables for $|L| = 2, 3$, and the following formulas give the other 12 cases.

Define the five-variable base table m_0 by

```
0,0,0,0,0,1,0,5,0,1,0,5,2,3,5,10,
0,1,0,5,3,4,5,10,0,5,5,10,5,10,10,17.
```

For $|L| \le 1$, set

$$m_L(S) = m_0(S \cap N) + m_0(b_L^{-1}(S \cap N'_L)).$$

For $L = N$, set $m_L = m_0$. For $|L| = 4$, write $N \setminus L = \{i\}$, let i' be its copied label, put $d_i = m_0(N) - m_0(L)$, and define

$$m_L(S) = m_0\big((S \cap N) \cup \{i : i' \in S\}\big) + d_i\, \mathbf{1}_{\{i,i'\} \subseteq S}.$$

The 20 printed tables are independently scaled integer witnesses; there is no requirement that their restrictions be equal across different overlaps.

Lemma 2. *For all 32 overlaps, the functions in* (2) *are polymatroids, induce exactly Table 1 on both copies, and satisfy* $A_L(g_L) = 0$.

Proof. This is a finite exact certificate proof. All coordinates are given above or in Appendix B. Substitution verifies $g_L(\varnothing) = 0$ and the following elementary inequalities for every admissible S, i, j:

$$\begin{aligned} g_L(S) &\le g_L(Si), && i \notin S, \\ g_L(S) + g_L(Sij) &\le g_L(Si) + g_L(Sj), && i, j \notin S,\ i \ne j. \end{aligned}$$

The first family implies monotonicity by adding missing elements one at a time. The second family implies diminishing returns: successively adding the members of $T \setminus S$ gives

$$g_L(Ti) - g_L(T) \le g_L(Si) - g_L(S) \quad (S \subseteq T,\ i \notin T).$$

Inducting on $A \setminus B$ and adding these increment inequalities proves full submodularity for every pair A, B. This argument is proved generically in `LocalToGlobal.lean`; `LocalPM.lean` transports it through the proved mask/finite-set encoding. Thus the formal conclusion is the full polymatroid definition, not a replacement by unchecked local conditions.

Evaluating all 80 elementary CMIs on each copy gives zeros exactly at Table 1, and positive integers everywhere else. The four-coordinate expression $A_L(g_L)$ is zero in every case. The full polymatroid property, both exact CI models, and the adhesion equality are separately kernel-checked from the integer data. The empty overlap is proved by an algebraic direct sum of two base polymatroids, while the full overlap includes the identity copy.

There is an additional exact ordinary certificate observation: for each fixed L and every $S \subseteq N$, direct substitution gives $g_L(S) = g_L(b_L(S))$. This numerical equality within each overlap is checked in the supplied integer data; it is not part of the Lean endpoint or the stated theorem.

The independent standard-library checker additionally enumerates every unordered subset pair, without the local-to-global reduction: 1,603,888 submodularity checks and 248,832 comparable-pair monotonicity checks pass. That checker is supplementary to the Lean proof, not an axiom used by it. □

Lemma 3. *If A, B form a modular pair in a polymatroid g, then for every $A \cap B \subseteq S \subseteq A \cup B$,*

$$g(S) = g(S \cap A) + g(S \cap B) - g(A \cap B). \tag{3}$$

Consequently every crossing elementary CMI, with conditioning containing $A \cap B$, vanishes.

Proof. Submodularity of $S \cap A, S \cap B$ gives one inequality in (3). Add submodularity for A, S and for $B, A \cup S$. Because $A \cap B \subseteq S \subseteq A \cup B$, the resulting inequality is

$$g(A) + g(B) + g(S) \geq g(S \cap A) + g(S \cap B) + g(A \cup B).$$

Use $g(A) + g(B) = g(A \cup B) + g(A \cap B)$ for the reverse inequality. For $P \subseteq A \setminus B$, $Q \subseteq B \setminus A$ and an admissible K, apply (3) to K, KP, KQ, KPQ. The terms cancel, giving $g(KP) + g(KQ) - g(K) - g(KPQ) = 0$. Taking P, Q to be singletons yields the required elementary statements. The stronger arbitrary-subset statement is formalized. □

Any other bijective L-copy is obtained by relabeling the canonical union. Pull back subsets under that ground-set bijection. This preserves every coordinate, normalization, monotonicity, submodularity, CI value, and the modular-pair equality. `Transport.lean` constructs actual powerset encodings and proves this for every ground-set type and bijection. Lemmas 2 and 3 therefore prove part 1 of Theorem 1 with its actual copy quantifiers.

3 Negative proof: no common real profile exists

Assume that h is a real polymatroid inducing exactly $\mathcal{M}$ and is numerically selfadhesive. Choose numerical gluings g_{34} and g_{145} at $L = \{3, 4\}$ and $L = \{1, 4, 5\}$. Their original and copied restrictions are h itself. The two copied ground sets use the canonical labeling from Section 2.

Let $Q(h, g_{34}, g_{145})$ be the positive integer combination of CMIs in Appendix A. It contains five base CMIs and 46 CMIs in the two copied polymatroids. Every summand is nonnegative. Direct expansion gives the exact identity

$$I_h(2; 4 \mid 135) + Q = R(h) + 3A_{34}(g_{34}) + 3A_{145}(g_{145}), \tag{4}$$

where

$$\begin{aligned} R(h) = {} & 3I_h(1; 2 \mid 45) + 2I_h(1; 3 \mid 4) + I_h(1; 3 \mid 5) + I_h(1; 3 \mid 24) \\ & + 2I_h(1; 4 \mid 3) + I_h(1; 4 \mid 25) + I_h(1; 5 \mid 3) \\ & + I_h(2; 4) + I_h(2; 5) + 2I_h(2; 5 \mid 134) \\ & + 2I_h(3; 4 \mid 12) + 2I_h(4; 5 \mid 23). \end{aligned}$$

To check (4), replace each CMI by its four entropy coordinates, substitute the equal numerical copy restrictions, and collect coefficients. Every residual coefficient is zero. No normalization of $h(N)$, no rationality hypothesis, and no chosen ray decomposition is used.

Each CMI in $R(h)$ is prescribed zero. Both adhesive gaps vanish. Thus the right-hand side is zero, while every term on the left-hand side is nonnegative. In particular

$$I_h(2;4 \mid 135) = 0.$$

That statement is absent from Table 1; exact induction of $\mathcal{M}$ requires it to be positive. This contradiction proves part 2 of Theorem 1. The Lean proof instantiates all 51 submodular inequalities on actual real-valued functions and checks the coefficient identity by ring normalization. It does not assume a solver's infeasibility verdict or a Farkas certificate validator.

4 Information-theoretic consequence and prior work

The foundational consequence is precise. Boolean CI-level selfadhesion tests can all pass while no common numerical Shannon-polymatroid profile supports the corresponding copy constructions. Hence structural selfadhesivity forgets compatibility information needed by numerical copy tests. A solver that separately realizes the CI model for each overlap cannot conclude that there is one numerically selfadhesive realization.

This is an entropy-geometry and CI-inference result. It does not characterize the five-variable entropy cone or supply a channel capacity or coding bound. It also does not assert that the construction is a distribution: polymatroids are the Shannon outer relaxation.

Boege's four-variable computation reports 1,224 structurally selfadhesive isomorphism types, each induced by a numerically selfadhesive polymatroid [1]. The five-variable result therefore lies beyond that established positive case. Minimality relative to that published enumeration is contextual; this project does not Lean-formalize that earlier enumeration. The newer Boege–Bolt–Studený paper proves numerical selfadhesivity implies structural selfadhesivity and that almost-entropic polymatroids are numerically selfadhesive [2, Lemma 8.2]. Its five-variable extreme-ray computations cover a restricted family, rather than all CI faces.

There is significant adjacent prior work. The ScaleMath repository committed an exact five-variable counterexample on 23 August 2026 UTC [5]. Its stated theorem answers BBS Open Problem 3 by separating

$$\mathfrak{st}^{\diamond}(5) \subsetneq \mathfrak{sg}^{\diamond}(5) \cap \mathfrak{st}(5).$$

Its 20-CI model has no structural adhesion at one overlap. Our model has a structural adhesion at every overlap and separates the different inclusion

$$\{\mathcal{M}_h : h \text{ is numerically selfadhesive}\} \subsetneq \mathfrak{st}^{\diamond}(5).$$

The distinction is in the theorem's hypotheses and quantifiers, not merely in a different CI string. Searching higher-dimensional faces and proving exact CMI identities were already used there and are not claimed as new methods.

The present model's non-entropicness is also not new information. Studený's older implication (I:1), under $(X, Y, Z, U) = (4, 5, 3, 1)$, applies to the prescribed zeros $45 \mid \varnothing$, $45 \mid 3$, $13 \mid 4$, $13 \mid 5$ and forces $13 \mid \varnothing$ for actual distributions [4, Section V.B]. Our model omits that conclusion. This old observation does not settle whether there is a numerically selfadhesive polymatroid inducing the model. The contribution assessed here is the full structural/numerical separation.

5 Formal correspondence and reproducibility

The pinned toolchain is Lean 4.35.0-rc3, with mathlib commit

`0db6cb99b412983de54b59ecc1b0b2747542cda3.`

The manifest pins nine dependencies. The verification driver creates a fresh directory containing source files only and compiles every project module in dependency order. It reuses only pinned dependency build caches when requested; no project compiled object is reused. It checks dependency commits and tracked working-tree cleanliness and prints endpoint axioms.

The core public endpoints are:

```
InfoE3.counterexample
InfoE3.counterexample_all_labelings
InfoE3.structural_with_crossing_independences
```

`IndependentScopeAudit.lean` independently expands the full real-profile quantifier, all-subset polymatroid hypotheses, every overlap, and a literal list of the 26 zero positions. `Semantics.lean` proves the complete 80-statement catalog, exact copy embeddings, and their intersection and union. `AdhesiveSemantics.lean` and `Transport.lean` prove the semantic bridges used above.

The positive witnesses are integers cast to real numbers. The negative statement ranges over every real-valued profile. No formal theorem assumes that all profiles are rational, that a chosen three-ray family exhausts them, or that the search LP has returned a reliable answer. The whole-face mathematical obstruction is the explicit real-coordinate identity.

The package includes the complete Lean sources, pinned manifest, certificate JSON, standard-library exact checkers, source generators and the record of a fresh replay with its compilation logs and endpoint axiom output. The permitted foundational axiom set is precisely `propext`, `Classical.choice`, and `Quot.sound`. No admission, added axiom, or native-decide trust shortcut is used.

Reproduce from the package root after installing the pinned Lean toolchain:

```
python3 proof/E3_all_overlap_check.py \
  proof/E3_all_overlap_polymatroids.json \
  --output /tmp/e3-positive-check.json
python3 proof/E3_negative_check.py
python3 proof/E3_lean_witness_correspondence.py
python3 proof/verify_E3.py --lake lake
```

With an existing correctly pinned dependency checkout, add the following option to the last command:

```
--reuse-packages /absolute/path/to/.lake/packages
```

Discovery used floating-point LPs to propose witnesses, followed by exact reconstruction. Rechecking the theorem requires no optimization solver.

6 Limitations

The literature comparison covered primary arXiv versions and sources, journal versions, theses, MathOverflow, scholarly and code repositories, including other AI research repositories, and Lean

libraries. Some services and publisher pages were inaccessible, and searches that find nothing do not establish priority. Absence from the published restricted ray lists is not evidence of novelty either. A specialist in conditional independence and entropy regions should assess novelty and significance before any priority claim. The theorem and its formal verification stand regardless; an earlier equivalent result would change only the novelty assessment.

7 Research process and AI disclosure

This is fully autonomous agentic research. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed, no expert validation is claimed, and the novelty assessment is provisional.

The question, the construction, the certificates, the Lean formalization, the literature searches and an earlier version of this note were produced by autonomous AI research agents in an earlier session, using linear programming to discover the witnesses and exact reconstruction to certify them. The records of that session do not name the AI system used.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It compared the Lean definitions of polymatroids, copies, adhesion, the CI catalogue and both selfadhesivity notions with the statements above and with Question 4.18 and Definition 4.3 of [1], which it read in the arXiv version; it reran the exact positive and negative checkers and the fresh Lean replay described in Section 5, and it revised this note. On 1 November 2026 it also revised the front matter and references without changing the mathematics. AI checks are not human peer review.

Lean verification establishes only the formal statements described in Section 5; the limits of the literature comparison are stated in Section 6.

A Complete separating identity data

In Table 2, Q is the sum of each row's coefficient times $I_f(i; j \mid K)$. Labels in a copied profile are the local canonical labels $1, \ldots, 10 - |L|$. Thus for g_{34} the copies of $1, 2, 5$ have labels $6, 7, 8$; for g_{145} the copies of $2, 3$ have labels $6, 7$. Empty conditioning is written $\varnothing$. Rows labeled h use the original profile directly. All coefficients are positive integers. Together with the displayed $R(h)$, this specifies every term of (4).

Table 2: The 51 nonnegative terms in Q.

Row	Profile	i	$j \mid K$	Coefficient
1	h	1	$2 \mid \varnothing$	2
2	h	1	$3 \mid \{2,5\}$	1
3	h	1	$4 \mid \{2,3\}$	2
4	h	3	$4 \mid \{2,5\}$	1
5	h	4	$5 \mid \{3\}$	1
6	g_{34}	1	$3 \mid \{5,6\}$	1
7	g_{34}	1	$6 \mid \{3\}$	1
8	g_{34}	1	$6 \mid \{4\}$	1
9	g_{34}	1	$6 \mid \{2,3,4,5\}$	1
10	g_{34}	1	$7 \mid \{3,4,6\}$	1

Row	Profile	i	$j \mid K$	Coefficient
11	g_{34}	1	$7 \mid \{3,4,5,6,8\}$	1
12	g_{34}	1	$7 \mid \{2,3,4,5,6,8\}$	1
13	g_{34}	1	$8 \mid \{3,4,5,6\}$	1
14	g_{34}	1	$8 \mid \{2,3,4,5,6\}$	1
15	g_{34}	1	$8 \mid \{3,4,5,6,7\}$	1
16	g_{34}	2	$6 \mid \{1,3,5\}$	1
17	g_{34}	2	$6 \mid \{1,3,4,7\}$	1
18	g_{34}	2	$6 \mid \{3,4,5,7\}$	1
19	g_{34}	2	$7 \mid \{3,4,5\}$	1
20	g_{34}	2	$7 \mid \{1,3,4,5\}$	1
21	g_{34}	2	$7 \mid \{1,3,4,5,6,8\}$	1
22	g_{34}	2	$8 \mid \{3,4,6\}$	1
23	g_{34}	2	$8 \mid \{1,3,4,5,6\}$	1
24	g_{34}	2	$8 \mid \{1,3,4,5,6,7\}$	1
25	g_{34}	3	$4 \mid \{1,6\}$	1
26	g_{34}	3	$6 \mid \{4,5\}$	1
27	g_{34}	4	$5 \mid \{6\}$	1
28	g_{34}	4	$6 \mid \{1,2,3,5\}$	1
29	g_{34}	5	$6 \mid \{1\}$	1
30	g_{34}	5	$6 \mid \{3\}$	1
31	g_{34}	5	$6 \mid \{1,2,3,4,7\}$	1
32	g_{34}	5	$7 \mid \{1,3,4\}$	1
33	g_{34}	5	$7 \mid \{3,4,6,8\}$	2
34	g_{34}	5	$8 \mid \{3,4,6\}$	2
35	g_{34}	5	$8 \mid \{2,3,4,6,7\}$	1
36	g_{34}	7	$8 \mid \{2,3,4,6\}$	1
37	g_{145}	1	$2 \mid \{4,5,6,7\}$	1
38	g_{145}	1	$5 \mid \{2,4,6\}$	1
39	g_{145}	1	$7 \mid \{2,4\}$	1
40	g_{145}	2	$5 \mid \{1,4,6,7\}$	1
41	g_{145}	2	$6 \mid \{1,4\}$	1
42	g_{145}	2	$6 \mid \{4,5\}$	1
43	g_{145}	2	$6 \mid \{1,4,7\}$	1
44	g_{145}	2	$7 \mid \{4,5\}$	1
45	g_{145}	2	$7 \mid \{4,6\}$	1
46	g_{145}	2	$7 \mid \{1,3,4,5,6\}$	1
47	g_{145}	3	$6 \mid \{1,2,4,5\}$	1
48	g_{145}	3	$6 \mid \{1,2,4,5,7\}$	2
49	g_{145}	3	$7 \mid \{1,2,4,5\}$	2
50	g_{145}	3	$7 \mid \{1,4,5,6\}$	1
51	g_{145}	5	$6 \mid \{2,4,7\}$	1

B All nonautomatic integer witness tables

The following arrays give $m_L(S)$ for every mask $S = 0, \ldots, 2^{10-|L|} - 1$. Each line begins with its starting mask followed by 16 successive values. The displayed data, the base table and formulas in Section 2, and (2) determine all 7,776 coordinates of all 32 witnesses. No hidden solver output or external data file is needed to reconstruct them. The JSON file is an equivalent machine-readable form.

Overlap $L = \{1, 2\}$ (mask 3), ground size 8

```
  0: 0,0,0,0,0,3,0,15,0,3,0,15,6,9,15,30
 16: 0,3,0,15,9,12,15,30,0,15,15,30,15,30,30,51
 32: 0,3,0,15,0,13,9,30,1,12,9,30,11,24,24,45
 48: 2,9,9,30,11,24,24,45,9,24,24,45,24,45,43,66
 64: 0,3,0,15,0,6,9,30,4,9,9,30,10,21,24,45
 80: 0,9,9,30,9,21,24,45,7,24,24,45,22,42,43,66
 96: 6,9,15,30,8,21,24,45,10,21,24,45,21,36,39,60
112: 12,21,24,45,21,36,39,60,19,36,39,60,34,57,58,81
128: 0,3,0,15,0,9,9,30,0,9,9,30,7,24,24,45
144: 2,9,9,30,11,24,24,45,9,24,24,45,24,45,43,66
160: 9,12,15,30,9,24,24,45,12,24,24,45,22,39,39,60
176: 11,24,24,45,20,39,39,60,24,39,39,60,39,60,58,81
192: 0,15,15,30,4,21,24,45,6,21,24,45,15,36,39,60
208: 4,21,24,45,15,36,39,60,19,36,39,60,34,57,58,81
224: 15,30,30,51,19,42,41,66,21,42,41,66,32,57,56,81
240: 21,42,41,66,32,57,56,81,36,57,56,81,51,78,75,102
```

Overlap $L = \{1, 3\}$ (mask 5), ground size 8

```
  0: 0,0,0,0,0,4,0,20,0,4,0,20,8,12,20,40
 16: 0,4,0,20,12,16,20,40,0,20,20,40,20,40,40,68
 32: 0,0,0,11,0,20,8,36,0,8,8,31,8,28,28,56
 48: 0,10,8,31,12,32,28,56,11,31,31,54,31,56,51,84
 64: 0,4,0,8,8,12,8,28,4,8,8,28,16,20,28,48
 80: 4,8,8,28,20,24,28,48,8,28,28,48,28,48,48,76
 96: 0,20,8,31,20,40,28,56,8,28,26,51,28,48,48,76
112: 10,30,26,51,32,52,48,76,29,51,49,74,51,76,71,104
128: 0,4,0,10,12,16,12,32,4,8,10,30,20,24,32,52
144: 5,9,10,30,24,28,32,52,10,30,30,52,32,52,52,80
160: 0,20,8,31,20,40,28,56,8,28,26,51,28,48,48,76
176: 10,30,26,51,32,52,48,76,29,51,49,74,51,76,71,104
192: 0,20,11,31,20,40,31,56,8,28,29,51,28,48,51,76
208: 10,30,29,51,32,52,51,76,31,51,54,76,53,76,76,104
224: 20,40,31,56,40,68,51,84,28,48,49,76,48,76,71,104
240: 30,52,49,76,52,80,71,104,52,74,75,102,76,104,99,132
```

Overlap $L = \{2, 3\}$ (mask 6), ground size 8

```
  0: 0,0,0,0,0,6,0,30,0,6,0,30,12,18,30,60
 16: 0,6,0,30,18,24,30,60,0,30,30,60,30,60,60,102
 32: 0,0,0,6,6,15,30,60,0,6,6,36,22,32,60,90
 48: 0,6,6,36,25,34,60,90,0,30,36,66,44,74,90,132
 64: 0,0,0,6,12,18,30,60,0,6,6,36,28,36,60,90
 80: 0,6,6,36,30,36,60,90,0,30,36,66,46,78,90,132
 96: 6,6,30,36,18,28,60,90,6,18,36,66,38,50,90,120
112: 6,12,36,66,40,50,90,120,12,42,66,96,60,92,120,162
```

```
128: 0,0,0,6,18,24,30,60,0,8,6,36,30,38,60,90
144: 5,12,6,36,37,44,60,90,6,36,36,66,50,80,90,132
160: 6,6,30,36,24,34,60,90,12,22,36,66,42,52,90,120
176: 11,20,36,66,44,54,90,120,20,50,66,96,64,94,120,162
192: 0,0,30,36,30,36,60,90,6,14,36,66,46,56,90,120
208: 6,13,36,66,49,56,90,120,12,42,66,96,66,98,120,162
224: 30,30,60,72,60,70,102,132,36,48,72,102,80,92,132,162
240: 39,48,72,102,82,92,132,162,48,78,102,132,102,134,162,204
```

Overlap $L = \{1, 2, 3\}$ (mask 7), ground size 7

```
  0: 0,0,0,0,0,2,0,7,0,2,0,7,3,5,7,14
 16: 0,2,0,7,4,6,7,14,0,7,7,14,7,14,14,23
 32: 0,2,0,7,3,5,7,14,0,4,7,14,6,10,14,21
 48: 2,6,7,14,7,11,14,21,4,11,14,21,11,19,21,30
 64: 0,2,0,7,4,6,7,14,0,4,7,14,7,11,14,21
 80: 4,6,7,14,8,12,14,21,4,11,14,21,11,20,21,30
 96: 0,7,7,14,7,14,14,23,4,11,14,21,11,19,21,30
112: 6,13,14,21,13,20,21,30,10,18,21,29,17,28,28,39
```

Overlap $L = \{1, 4\}$ (mask 9), ground size 8

```
  0: 0,0,0,0,0,1,0,5,0,1,0,5,2,3,5,10
 16: 0,1,0,5,3,4,5,10,0,5,5,10,5,10,10,17
 32: 0,0,0,3,0,2,3,8,0,5,3,9,2,7,8,14
 48: 0,3,3,8,3,7,8,13,3,9,8,14,8,14,14,21
 64: 0,1,0,2,1,2,2,7,2,3,2,7,4,5,7,12
 80: 0,2,2,7,4,6,7,12,2,7,7,12,7,12,12,19
 96: 0,5,3,8,2,7,8,13,5,10,8,14,7,12,13,19
112: 3,8,8,13,7,12,13,18,8,14,13,19,13,19,19,26
128: 0,1,0,3,0,2,3,8,0,5,3,9,2,7,8,14
144: 2,4,3,8,5,7,8,13,3,9,8,14,8,14,14,21
160: 0,5,3,8,2,7,8,13,5,10,8,14,7,12,13,19
176: 3,8,8,13,7,12,13,18,8,14,13,19,13,19,19,26
192: 3,4,4,7,4,6,7,12,5,10,8,14,7,12,13,19
208: 5,7,7,12,9,11,12,17,8,14,13,19,13,19,19,26
224: 5,10,8,13,7,12,13,18,10,17,14,21,12,19,19,26
240: 8,13,13,18,12,17,18,23,14,21,19,26,19,26,26,33
```

Overlap $L = \{2, 4\}$ (mask 10), ground size 8

```
  0: 0,0,0,0,0,27,0,102,0,27,0,102,43,70,102,204
 16: 0,27,0,102,59,86,102,204,0,102,102,204,102,204,204,338
 32: 0,0,0,63,0,45,63,165,27,82,102,204,70,146,204,306
 48: 0,60,63,165,59,132,165,267,55,157,204,306,157,280,306,440
 64: 0,13,0,63,0,40,63,165,43,97,102,204,92,146,204,306
 80: 0,63,63,165,59,132,165,267,70,172,204,306,172,280,306,440
 96: 27,40,102,165,40,85,165,267,70,152,204,306,134,222,306,408
112: 27,100,165,267,114,187,267,369,125,227,306,408,227,356,408,542
128: 0,31,0,63,0,58,63,165,0,82,102,204,64,146,204,306
144: 0,80,63,165,59,139,165,267,55,157,204,306,157,280,306,440
160: 27,58,102,165,40,122,165,267,102,184,204,306,166,248,306,408
176: 27,118,165,267,118,209,267,369,157,259,306,408,259,382,408,542
192: 59,90,102,165,59,117,165,267,102,184,204,306,166,248,306,408
208: 59,150,165,267,118,219,267,369,157,259,306,408,259,382,408,542
224: 86,117,204,267,99,187,267,369,204,286,338,440,268,356,440,542
240: 97,188,267,369,188,289,369,471,259,361,440,542,361,490,542,676
```

Overlap $L = \{1, 2, 4\}$ (mask 11), ground size 7

```
  0: 0,0,0,0,0,4,0,11,0,4,0,11,5,9,11,22
 16: 0,4,0,11,6,10,11,22,0,11,11,22,11,22,22,35
 32: 0,4,0,11,0,8,11,22,5,9,11,22,10,18,22,33
 48: 2,8,11,22,8,18,22,33,7,18,22,33,18,31,33,46
 64: 0,4,0,11,0,8,11,22,0,11,11,22,7,18,22,33
 80: 2,8,11,22,8,18,22,33,7,18,22,33,18,31,33,46
 96: 6,10,11,22,7,18,22,33,11,22,22,35,18,31,33,46
112: 8,19,22,33,18,31,33,46,18,31,33,46,29,44,44,59
```

Overlap $L = \{3, 4\}$ (mask 12), ground size 8

```
  0: 0,0,0,0,0,1,0,5,0,1,0,5,2,3,5,10
 16: 0,1,0,5,3,4,5,10,0,5,5,10,5,10,10,17
 32: 0,0,0,1,1,2,1,6,1,2,1,6,3,4,6,11
 48: 0,1,1,6,4,5,6,11,1,6,6,11,6,11,11,18
 64: 0,0,0,2,0,1,2,7,0,1,2,7,5,6,8,13
 80: 0,3,2,7,3,6,7,12,2,7,7,13,8,13,13,20
 96: 0,1,1,6,5,6,7,12,5,6,7,12,10,11,13,18
112: 2,5,6,11,8,11,12,17,7,12,12,18,13,18,18,25
128: 0,0,0,2,3,4,3,8,0,1,2,7,5,6,8,13
144: 0,1,2,7,6,7,8,13,2,7,7,13,8,13,13,20
160: 1,1,1,6,4,5,7,12,5,6,7,12,10,11,13,18
176: 1,5,6,11,7,11,12,18,7,12,12,18,13,18,18,25
192: 0,1,2,6,5,6,7,12,5,6,7,12,10,11,13,18
208: 2,5,6,11,8,11,12,17,7,12,12,18,13,18,18,25
224: 5,6,7,12,10,11,13,18,10,11,13,18,17,18,20,25
240: 7,11,12,17,13,17,18,24,13,18,18,25,20,25,25,32
```

Overlap $L = \{1, 3, 4\}$ (mask 13), ground size 7

```
  0: 0,0,0,0,0,1,0,5,0,1,0,5,2,3,5,10
 16: 0,1,0,5,3,4,5,10,0,5,5,10,5,10,10,17
 32: 0,0,0,1,0,5,0,9,0,5,0,9,5,10,8,17
 48: 0,2,0,6,3,10,5,16,0,9,5,14,8,17,13,24
 64: 0,1,0,1,3,4,3,9,0,5,0,9,5,10,8,17
 80: 0,3,0,7,6,9,8,16,0,9,5,15,8,17,13,24
 96: 0,5,0,6,5,10,5,16,5,10,5,14,10,17,13,24
112: 0,7,2,12,8,16,10,23,5,14,10,20,13,24,18,31
```

Overlap $L = \{2, 3, 4\}$ (mask 14), ground size 7

```
  0: 0,0,0,0,0,2,0,7,0,2,0,7,3,5,7,14
 16: 0,2,0,7,4,6,7,14,0,7,7,14,7,14,14,23
 32: 0,2,0,7,2,4,7,14,2,4,7,14,5,11,14,21
 48: 2,4,7,14,8,11,14,21,4,11,14,21,11,20,21,30
 64: 0,2,0,7,4,8,7,14,0,4,7,14,7,11,14,21
 80: 2,6,7,14,8,12,14,21,4,11,14,21,11,20,21,30
 96: 2,4,7,14,6,11,14,21,7,11,14,21,14,20,23,30
112: 6,11,14,21,12,20,21,30,11,20,21,30,20,29,30,39
```

Overlap $L = \{1, 5\}$ (mask 17), ground size 8

```
  0: 0,0,0,0,0,1,0,5,0,1,0,5,2,3,5,10
 16: 0,1,0,5,3,4,5,10,0,5,5,10,5,10,10,17
 32: 0,0,0,2,0,2,2,7,0,2,2,7,3,7,7,14
 48: 0,5,3,9,3,8,8,14,3,9,8,14,8,14,13,21
 64: 0,1,0,1,1,2,1,6,0,2,1,6,3,6,6,13
 80: 3,4,3,8,6,7,8,13,3,8,8,13,8,13,13,20
 96: 0,5,1,7,2,7,6,12,2,7,6,12,7,12,11,19
112: 5,10,8,14,8,13,13,19,8,14,13,19,13,19,18,26
128: 0,1,0,2,0,3,2,7,0,3,2,7,4,7,7,14
144: 0,5,3,9,3,8,8,14,3,9,8,14,8,14,13,21
160: 0,5,1,7,2,7,6,12,2,7,6,12,7,12,11,19
176: 5,10,8,14,8,13,13,19,8,14,13,19,13,19,18,26
192: 2,3,2,7,3,6,6,12,3,7,6,12,8,12,11,19
208: 5,10,8,14,8,13,13,19,8,14,13,19,13,19,18,26
224: 5,10,6,14,7,13,11,19,7,14,11,19,12,19,16,26
240: 10,17,13,21,13,20,18,26,13,21,18,26,18,26,23,33
```

Overlap $L = \{2, 5\}$ (mask 18), ground size 8

```
  0: 0,0,0,0,0,43,0,173,0,43,0,173,72,115,173,346
 16: 0,43,0,173,101,144,173,346,0,173,173,346,173,346,346,577
 32: 0,0,0,158,0,120,158,331,0,143,158,331,130,273,331,504
 48: 43,155,173,346,173,293,346,519,137,310,346,519,310,504,519,750
 64: 0,0,0,158,29,129,158,331,45,144,158,331,146,273,331,504
 80: 101,146,173,346,202,302,346,519,146,319,346,519,319,504,519,750
 96: 43,128,173,331,138,278,331,504,118,281,331,504,268,431,504,677
112: 144,283,346,519,311,456,519,692,283,456,519,692,456,667,692,923
128: 0,12,0,158,41,120,158,331,122,165,158,331,194,273,331,504
144: 0,137,173,346,142,293,346,519,137,310,346,519,310,504,519,750
160: 43,155,173,331,142,293,331,504,165,308,331,504,295,446,504,677
176: 173,310,346,519,315,466,519,692,310,483,519,692,483,677,692,923
192: 72,140,173,331,142,278,331,504,209,308,331,504,310,446,504,677
208: 173,310,346,519,315,466,519,692,310,483,519,692,483,677,692,923
224: 115,283,346,514,283,451,514,687,283,451,514,687,451,662,687,908
240: 346,514,577,750,519,687,750,923,514,687,750,923,687,898,923,1154
```

Overlap $L = \{1, 2, 5\}$ (mask 19), ground size 7

```
  0: 0,0,0,0,0,3,0,9,0,3,0,9,4,7,9,18
 16: 0,3,0,9,5,8,9,18,0,9,9,18,9,18,18,29
 32: 0,3,0,9,2,6,9,18,0,6,9,18,6,14,18,27
 48: 5,8,9,18,10,15,18,27,6,15,18,27,15,25,27,38
 64: 0,3,0,9,2,10,9,18,3,6,9,18,7,15,18,27
 80: 0,9,9,18,7,16,18,27,6,15,18,27,15,26,27,38
 96: 4,7,9,18,7,15,18,27,8,14,18,27,15,23,27,36
112: 9,18,18,29,16,26,27,38,15,25,27,38,24,36,36,49
```

Overlap $L = \{3, 5\}$ (mask 20), ground size 8

```
  0: 0,0,0,0,0,1,0,5,0,1,0,5,2,3,5,10
 16: 0,1,0,5,3,4,5,10,0,5,5,10,5,10,10,17
 32: 0,0,0,1,1,2,1,6,0,1,1,6,3,4,6,11
 48: 1,2,1,6,4,5,6,11,1,6,6,11,6,11,11,18
 64: 0,0,0,0,0,1,1,6,0,1,0,5,2,5,6,12
```

```
 80: 0,1,1,6,5,6,7,12,1,6,6,12,7,12,12,19
 96: 0,1,1,5,5,6,6,11,1,4,5,10,7,10,11,17
112: 5,6,6,11,10,11,12,17,6,11,11,17,12,17,17,24
128: 0,0,0,1,2,3,2,7,0,1,1,6,4,5,7,12
144: 0,1,1,6,5,6,7,12,1,6,6,12,7,12,12,19
160: 1,1,2,4,3,4,5,10,1,5,4,10,5,9,10,16
176: 5,6,6,11,10,11,12,17,6,11,11,17,12,17,17,24
192: 0,1,1,5,5,6,6,11,1,4,5,10,7,10,11,17
208: 5,6,6,11,10,11,12,17,6,11,11,17,12,17,17,24
224: 5,6,6,10,10,11,12,17,6,10,10,16,12,16,17,23
240: 10,11,12,17,17,18,19,24,12,17,17,23,19,24,24,31
```

Overlap $L = \{1, 3, 5\}$ (mask 21), ground size 7

```
  0: 0,0,0,0,0,3,0,15,0,3,0,15,6,9,15,30
 16: 0,3,0,15,9,12,15,30,0,15,15,30,15,30,30,51
 32: 0,0,0,8,0,15,1,28,0,9,3,25,9,24,18,45
 48: 0,15,5,28,15,30,21,48,4,27,20,48,25,48,41,69
 64: 0,3,0,7,6,9,6,22,0,6,4,22,12,18,21,39
 80: 0,15,0,27,15,30,21,48,4,27,19,47,25,48,40,69
 96: 0,15,1,23,15,30,16,43,9,24,16,40,24,39,33,60
112: 15,30,20,48,30,51,41,69,24,47,40,68,45,69,61,90
```

Overlap $L = \{2, 3, 5\}$ (mask 22), ground size 7

```
  0: 0,0,0,0,0,5,0,25,0,5,0,25,10,15,25,50
 16: 0,5,0,25,15,20,25,50,0,25,25,50,25,50,50,85
 32: 0,9,0,25,5,27,25,50,0,22,25,50,18,40,50,75
 48: 5,19,25,50,20,43,50,75,18,43,50,75,43,75,75,110
 64: 0,5,0,25,10,15,25,50,0,14,25,50,24,38,50,75
 80: 0,14,25,50,25,39,50,75,14,39,50,75,39,73,75,110
 96: 5,14,25,50,15,38,50,75,14,39,50,75,38,63,75,100
112: 25,39,50,75,50,73,85,110,39,64,75,100,73,107,110,145
```

Overlap $L = \{4, 5\}$ (mask 24), ground size 8

```
  0: 0,0,0,0,0,5,0,25,0,5,0,25,10,15,25,50
 16: 0,5,0,25,15,20,25,50,0,25,25,50,25,50,50,85
 32: 0,0,0,0,0,5,0,25,5,10,6,31,16,21,31,56
 48: 5,10,5,30,20,25,30,55,25,50,50,75,50,75,75,110
 64: 0,0,0,2,0,5,2,27,0,5,2,27,11,17,27,52
 80: 0,5,2,27,15,24,27,54,25,50,50,75,50,75,75,110
 96: 0,5,2,14,4,10,14,39,25,30,27,52,36,42,52,77
112: 25,30,27,52,40,49,52,79,50,75,75,100,75,100,100,135
128: 0,2,0,2,2,7,2,27,10,15,12,37,21,27,37,62
144: 15,20,15,40,30,35,40,65,25,50,50,75,50,75,75,110
160: 5,7,5,7,7,12,7,32,15,21,18,43,27,33,43,68
176: 20,25,20,45,35,40,45,70,50,75,75,100,75,100,100,135
192: 0,4,2,14,4,9,14,39,25,30,27,52,36,42,52,77
208: 25,30,27,52,40,49,52,79,50,75,75,100,75,100,100,135
224: 25,30,28,40,29,35,40,65,50,56,53,78,62,68,78,103
240: 50,56,53,78,65,75,78,105,85,110,110,135,110,135,135,170
```

Overlap $L = \{1, 4, 5\}$ (mask 25), ground size 7

```
  0: 0,0,0,0,0,4,0,11,0,4,0,11,5,9,11,22
 16: 0,4,0,11,6,10,11,22,0,11,11,22,11,22,22,35
 32: 0,0,0,7,0,7,7,20,0,11,7,18,7,18,18,31
 48: 0,11,7,18,7,18,18,31,11,22,22,33,22,33,33,46
 64: 0,4,0,7,3,8,7,20,5,9,7,18,12,18,18,31
 80: 6,10,7,18,13,19,18,31,11,22,22,33,22,33,33,46
 96: 0,11,7,20,7,20,18,33,11,22,18,31,18,31,29,44
112: 11,22,18,31,18,31,29,44,22,35,33,46,33,46,44,59
```

Overlap $L = \{2, 4, 5\}$ (mask 26), ground size 7

```
  0: 0,0,0,0,0,2,0,7,0,2,0,7,3,5,7,14
 16: 0,2,0,7,4,6,7,14,0,7,7,14,7,14,14,23
 32: 0,3,0,7,0,5,7,14,2,7,7,14,7,12,14,21
 48: 2,7,7,14,7,12,14,21,7,14,14,21,14,21,21,30
 64: 0,3,0,7,3,6,7,14,3,7,7,14,7,12,14,21
 80: 4,8,7,14,9,13,14,21,7,14,14,21,14,21,21,30
 96: 2,8,7,14,5,12,14,21,5,12,14,21,12,19,21,28
112: 6,13,14,21,12,19,21,28,14,21,23,30,21,28,30,39
```

Overlap $L = \{3, 4, 5\}$ (mask 28), ground size 7

```
  0: 0,0,0,0,0,5,0,22,0,5,0,22,9,14,22,44
 16: 0,5,0,22,13,18,22,44,0,22,22,44,22,44,44,74
 32: 0,0,0,14,5,19,10,36,5,17,14,36,14,36,36,66
 48: 5,19,14,36,18,38,36,64,22,44,44,66,44,66,66,96
 64: 0,9,0,12,0,14,8,34,0,14,14,36,22,36,36,66
 80: 0,14,14,36,22,36,36,62,22,44,44,66,44,66,66,96
 96: 0,14,14,34,22,42,36,62,22,36,36,66,44,66,66,96
112: 22,36,36,64,44,64,64,92,44,66,66,96,74,96,96,126
```

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