Sharp root log-concavity of alternating Motzkin sums

Dev D. Goyal

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Abstract

Let M_n be the Motzkin numbers and let $T_n = \sum_{j=0}^{n}(-1)^{n-j}M_j$. The sequence $T_n^{1/n}$ is strictly log-concave on its tail starting at $n = 5$. The inequality is reversed at center $n = 5$, so no earlier tail is log-concave. This proves Conjecture 1 of Zhao [6], which asks for log-concavity from index nine, with the sharper starting index five. The argument combines established ratio bounds with a size estimate. These estimates prove the infinite range, and exact integer comparisons cover the remaining 25 centers. The Lean formalization proves the real-root inequality and its sharp threshold, and it proves that the recursive sequence in the analytic proof equals the finite sum in the conjecture.

1 The problem and the result

A positive sequence (a_n) is log-concave if $a_n^2 \geq a_{n-1}a_{n+1}$ at each interior index. Strict log-concavity means that every such inequality is strict. For a tail starting at s, the interior indices are $n \geq s+1$. Thus the first inequality on a tail has its center one index after the start.

The Motzkin numbers satisfy the recurrence

$$M_0 = M_1 = 1, \qquad (n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1} \quad (n \geq 1). \tag{1}$$

These conditions determine the sequence uniquely over $\mathbb{Q}$ or $\mathbb{R}$. Define

$$T_n = \sum_{j=0}^{n}(-1)^{n-j}M_j. \tag{2}$$

The first values are

$$T_0, T_1, \ldots, T_9 = 1, 0, 2, 2, 7, 14, 37, 90, 233, 602.$$

Cancellation of consecutive sums gives

$$T_{n+1} = M_{n+1} - T_n. \tag{3}$$

Zhao studies this sequence in [6]. His Conjecture 1 asks whether $(T_n^{1/n})_{n\geq 9}$ is log-concave. The following theorem gives a sharper statement.

Theorem 1. *The values T_n are positive for $n \geq 2$. For every integer $n \geq 6$,*

$$T_n^{2(n^2-1)} > T_{n-1}^{n(n+1)}T_{n+1}^{n(n-1)}. \tag{4}$$

Thus $(T_n^{1/n})_{n\geq 5}$ is strictly log-concave. At center five the inequality is strictly reversed:

$$(T_5^{1/5})^2 < T_4^{1/4}T_6^{1/6}. \tag{5}$$

Consequently, five is the smallest starting index of a log-concave tail.

The proof uses known ratio bounds. The Motzkin bounds follow from the interlacing method of Došlić and Veljan [2]. Zhao proves an interlacing estimate for the alternating sums [6, eqs. (7)–(8)]. Proofs of the needed estimates are included. General methods for root log-concavity also appear in Chen, Guo, and Wang [1] and in Hou and Li [3]. The contribution is the alternating-sum inequality with its sharp threshold. The general method is due to the cited authors. A public manuscript added on 17 October 2026, before this paper, already claims strict root log-concavity at centers $n \geq 10$, that is, on Zhao's tail [5]. No claim of first priority is made.

The proof first bounds the logarithm of the ratio between the two sides of (4). The bound is positive for $n \geq 31$. Exact integer comparisons then establish the inequality for $6 \leq n \leq 30$. The Lean formalization proves Theorem 1, including the finite range and the sharp threshold; Section 5 states its scope.

2 Ratio and size estimates

Put

$$x_n = \frac{M_{n+1}}{M_n}, \qquad b_n = \frac{6(n+1)}{2n+5}.$$

The recurrence shows by induction that $M_n > 0$ for every n.

Lemma 2. *For every integer $n \geq 1$,*

$$b_n \leq x_n \leq b_{n+1}. \tag{6}$$

Moreover, $0 < T_n \leq M_n$ for $n \geq 2$.

Proof. At $n = 1$, the values satisfy $x_1 = 2 = b_2$ and $b_1 \leq 2$. For $n \geq 2$, define

$$f_n(t) = \frac{2n+3}{n+3} + \frac{3n}{(n+3)t} \quad (t > 0).$$

The recurrence gives $x_n = f_n(x_{n-1})$. The function f_n is decreasing. Thus the induction hypothesis gives

$$f_n(b_n) \leq x_n \leq f_n(b_{n-1}).$$

Direct calculation yields

$$f_n(b_n) - b_n = \frac{3(n-2)}{2(n+1)(n+3)(2n+5)} \geq 0,$$

and

$$b_{n+1} - f_n(b_{n-1}) = \frac{9}{2(n+3)(2n+7)} > 0.$$

This proves (6). Since $b_n > 1$ for $n \geq 1$, the Motzkin numbers increase strictly from index one. Now $T_1 = 0$ and $T_2 = 2 = M_2$. If $0 \leq T_{n-1} \leq M_{n-1}$ and $n \geq 2$, then

$$M_n - M_{n-1} \leq T_n = M_n - T_{n-1} \leq M_n.$$

The lower bound is positive. This proves the remaining claim by induction. □

Lemma 3. *Let $y_n = T_{n+1}/T_n$ for $n \geq 2$. Then*

$$x_n \leq y_n \leq x_{n+1} \qquad (n \geq 9). \tag{7}$$

Proof. Equation (3) gives

$$y_{n+1} = x_{n+1} - 1 + \frac{x_{n+1}}{y_n}. \tag{8}$$

At $n = 9$, the values are

$$x_9 = \frac{2188}{835}, \quad y_9 = \frac{793}{301}, \quad x_{10} = \frac{2899}{1094}.$$

The lower and upper bounds in (7) have positive gaps 3567/251335 and 5057/329294, respectively.

Suppose (7) holds at $n \geq 9$. Its upper bound and (8) imply $y_{n+1} \geq x_{n+1}$. For the other bound, put

$$f(t) = \frac{2n+5}{n+4} + \frac{3(n+1)}{(n+4)t}, \quad c = \frac{2n+7}{n+5}, \quad d = \frac{3(n+2)}{n+5},$$

and define

$$H(t) = f(t) - 1 + \frac{f(t)}{t} - c - \frac{d}{f(t)} \quad (t > 0).$$

The functions $f(t)$, $f(t)/t$, and $-d/f(t)$ are decreasing. For the middle function, write it as $A/t + B/t^2$ with $A, B > 0$. Thus H is decreasing. The recurrence and the induction hypothesis give

$$\begin{aligned} y_{n+1} - x_{n+2} &\leq x_{n+1} - 1 + \frac{x_{n+1}}{x_n} - x_{n+2} \\ &= H(x_n) \leq H(b_n) \\ &= -\frac{3(4n^2 - 13n - 53)}{4(n+1)(n+4)(n+5)(2n+5)} < 0. \end{aligned}$$

The numerator is positive for $n \geq 9$. This proves the induction step. □

Lemma 4. *For every integer $n \geq 0$,*

$$M_n \leq \frac{3^n}{n+1}. \tag{9}$$

Proof. The cases $n = 0, 1$ hold directly. For $n \geq 1$, apply the recurrence and the bounds at n and $n-1$ to obtain

$$M_{n+1} \leq \frac{3^n(3n+4)}{(n+1)(n+3)} < \frac{3^{n+1}}{n+2}.$$

The strict comparison follows from $3(n+1)(n+3) - (3n+4)(n+2) = 2n+1 > 0$. □

3 The infinite range

For $n \geq 3$, put $q_n = T_n/T_{n-1}$ and define

$$D_n = n(n^2-1)\left(\frac{2\log T_n}{n} - \frac{\log T_{n-1}}{n-1} - \frac{\log T_{n+1}}{n+1}\right).$$

All values in the logarithms are positive by Lemma 2. The factor $n(n^2-1)$ is positive. Hence $D_n > 0$ is equivalent both to strict root log-concavity at center n and to (4). Substitution gives

$$D_n = 2n \log q_n - n(n-1) \log \frac{q_{n+1}}{q_n} - 2 \log T_n. \tag{10}$$

For $n \geq 10$, Lemmas 2 and 3 give

$$q_n \geq b_{n-1} = \frac{3n}{n+3/2}, \qquad \frac{q_{n+1}}{q_n} \leq \frac{b_{n+2}}{b_{n-1}} = 1 + \frac{9}{n(2n+9)}.$$

Also, Lemmas 2 and 4 imply

$$\log T_n \leq n \log 3 - \log(n+1).$$

Use these estimates in (10). Since $\log(1+t) < t$ for $t > 0$,

$$\begin{aligned} D_n &\geq 2\log(n+1) - 2n \log\left(1 + \frac{3}{2n}\right) - n(n-1)\log\left(1 + \frac{9}{n(2n+9)}\right) \\ &> 2\log(n+1) - 3 - \frac{9(n-1)}{2n+9} =: F(n). \end{aligned} \tag{11}$$

For real $x \geq 10$,

$$F'(x) = \frac{8x^2 - 27x + 63}{(x+1)(2x+9)^2} > 0.$$

To give an exact positive value at the threshold, observe that

$$\log 2 = 2\int_0^{1/3} \frac{dt}{1-t^2} > 2\int_0^{1/3} (1+t^2)\, dt = \frac{56}{81}.$$

The strict inequality follows because $1/(1-t^2) > 1 + t^2$ for $0 < t < 1/3$. Consequently,

$$F(31) = 10\log 2 - 3 - \frac{270}{71} > \frac{560}{81} - 3 - \frac{270}{71} = \frac{637}{5751} > 0.$$

Monotonicity of F and (11) prove (4) for every integer $n \geq 31$.

4 The finite range and the sharp threshold

It remains to check the 25 centers $6 \leq n \leq 30$. Table 1 gives the values required for those comparisons, together with the value T_4 needed for sharpness. Every value follows from (1) and (3). Substitution in (4) gives a strict inequality at each of the 25 centers. These comparisons use integers only.

The following program reproduces the finite comparisons. It checks that each division in the recurrence is exact.

```
M = [1, 1]
for n in range(1, 31):
    numerator = (2*n + 3)*M[n] + 3*n*M[n-1]
    assert numerator % (n + 3) == 0
    M.append(numerator // (n + 3))
T = [sum((-1)**(n-j)*M[j] for j in range(n+1))
     for n in range(32)]
for n in range(6, 31):
    assert T[n]**(2*(n*n-1)) > (
        T[n-1]**(n*(n+1))*T[n+1]**(n*(n-1)))
assert T[5]**48 < T[4]**30*T[6]**20
```

At center five, the reverse comparison follows from

$$37^{10} - 2^{24}7^9 = 4131563191399737 > 0.$$

Indeed, $T_4 = 7$, $T_5 = 14$, and $T_6 = 37$. Squaring the inequality and multiplying by 7^{30} gives $14^{48} < 7^{30}37^{20}$. It is equivalent to (5). Every tail starting at an index at most four contains center five. The infinite argument and the finite comparisons prove Theorem 1.

n	T_n	n	T_n
4	7	18	4805596
5	14	19	13393688
6	37	20	37458331
7	90	21	105089228
8	233	22	295673995
9	602	23	834086420
10	1586	24	2358641377
11	4212	25	6684761124
12	11299	26	18985057352
13	30536	27	54022715450
14	83098	28	154000562759
15	227474	29	439742222070
16	625993	30	1257643249141
17	1730786	31	3602118427250

Table 1: Exact alternating Motzkin sums.

5 Formal verification and reproducibility

The Lean formalization defines rational Motzkin numbers through adjacent pairs. It defines the alternating sequence by (3). The source-definition module proves uniqueness for (1) and identifies the recursive sequence with (2). The resulting theorem therefore applies to the finite sum in Zhao's conjecture. Its assumptions specify only the recurrence and initial values. They do not include any estimate used in the proof.

The Challenge module states the problem and imports only Mathlib. It includes the existence of a sequence that satisfies the recurrence. The Solution module proves this statement, including existence and the sharp threshold.

Table 2 lists the principal declarations. Names are written relative to the namespace `MathAndTheory.Motzkin`.

Mathematical statement	Lean declaration
Source recurrence correspondence	`Source.motzkin_isMotzkin`; `Source.eq_motzkin_of_isMotzkin`
Source finite-sum correspondence	`Source.alternatingSum_eq_alternating`
Strict real-root inequality	`Source.finite_sum_roots_strict_log_concave`
Zhao's tail-nine conjecture	`Source.zhao_conjecture_one`
Center-five failure	`Source.finite_sum_roots_sharp_failure`
Optimal tail start	`Source.finite_sum_tail_start_optimal`
Complete problem statement	`Source.solution`

Table 2: Principal formal statements.

The proof modules establish the ratio estimates, positivity, the size bound, and the infinite range. Exact arithmetic produces proof terms for the remaining indices. The root function is the real power $T_n^{1/n}$.

The source-definition module passed its native Lean build and axiom audit. The source theorem uses only `propext`, `Classical.choice`, and `Quot.sound`. The repository's verification record gives the builds, statement comparisons, and independent kernel checks.

An independent Python program also counts Motzkin paths and applies the finite alternating sum directly. It agrees with the published OEIS table through index 300 [4]. It checks the exact inequality at centers 6 through 100 and the reverse comparison at center five. These

finite checks support the result. They do not prove the infinite range. The Lean statement uses the recurrence characterization. A formal proof of the lattice-path interpretation is outside its scope.

6 Research process and AI disclosure

This work is fully autonomous agentic research. Autonomous AI research agents produced the research, proofs, computations, formalization and drafts. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT, and GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities. This check is not expert validation or peer review.

Opus 5.5 by Anthropic used with Claude Code revised this manuscript editorially on 1 November 2026: it revised the abstract and this disclosure, added the earlier public claim [5] to the discussion of prior work, and completed the references. It did not change the mathematics.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed. No claim of first priority is made. Section 5 states the formal scope; the lattice-path interpretation of the Motzkin numbers is outside it.

References

[1] William Y. C. Chen, Jeremy J. F. Guo, and Larry X. W. Wang. Infinitely log-monotonic combinatorial sequences. *Advances in Applied Mathematics*, 52:99–120, 2014. doi: 10.1016/j.aam.2013.08.003. URL `https://arxiv.org/abs/1304.5160`.

[2] Tomislav Došlić and Darko Veljan. Logarithmic behavior of some combinatorial sequences. *Discrete Mathematics*, 308(11):2182–2212, 2008. doi: 10.1016/j.disc.2007.04.066. URL `https://arxiv.org/abs/math/0603379`.

[3] Qing-hu Hou and Zhongjie Li. Log-behavior of the root sequences of P-recursive sequences. *The Ramanujan Journal*, 70:14, 2026. doi: 10.1007/s11139-026-01371-z. URL `https://arxiv.org/abs/2310.19234`.

[4] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A187306, 2026. URL `https://oeis.org/A187306`. Accessed 26 October 2026.

[5] Public manuscript. Motzkin alternating sums: strict log-concavity of the root sequence, 2026. URL `https://github.com/xzwwwwww/math/commit/2e173134a5da24512ff9d2fbf1504b5634d29af8`. Public manuscript with a Chinese title (English description given here), file `motzkin_root_log_concavity.tex` in the xzwwwwww/math repository, added on 17 October 2026 in commit 2e173134a5da24512ff9d2fbf1504b5634d29af8. It claims strict root log-concavity at centers $n \geq 10$ by an integral method; its steps and publication status are not independently certified here.

[6] Feng-Zhen Zhao. The log-balancedness of the sequence for the alternating sums of Motzkin numbers. *INTEGERS*, 26:A55, 2026. doi: 10.5281/zenodo.19949645. URL `https://math.colgate.edu/~integers/aa55/aa55.pdf`.