Sharp log-concavity for alternating Motzkin sums

A bounded-forcing theorem, exact parameter thresholds, and an auditable proof package

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Abstract

Zhao asked whether the root sequence of the alternating sums T_n of the Motzkin numbers is log-concave from index nine [Z, Conjecture 1, p. 8]. This note proves a bounded-forcing theorem: real sequences that satisfy the Motzkin-type recurrence $(m+2)a_m = m(2a_{m-1}+3a_{m-2})+\varepsilon_m$ with errors $|\varepsilon_m| \le 2$, and a suitable seed at $n = 12$, grow, stay in an explicit quotient interval and are ratio log-concave with margin $800a_n^4/(n+3)^3$ for all $n \ge 12$. A corrected scaled-forcing corollary covers errors bounded by any $B \ge 0$. As applications, the alternating Motzkin sums have optimal quotient-tail start 10 and optimal root-tail start 5, and the sharp tail starts are classified for every member of the family $R_{n+1} + s(-1)^n$, $0 \le s \le 1$, which joins the shifted Riordan numbers to the alternating Motzkin sums. The bounded-forcing theorem (Theorem 2) and the scaled-forcing corollary have complete Lean proofs for arbitrary real sequences. The concrete-sequence, uniform-family, root-transfer and threshold-classification claims have conventional proofs, supported by exact arithmetic checks, and are outside that formalization. A public manuscript dated 17 October 2026 already claims the original root-log-concavity conjecture [P]. No first-priority claim is made for its resolution or for the extensions below, and a broader novelty assessment remains open.

1. Definitions and results

Define the Motzkin numbers by $M_0 = M_1 = 1$ and

$$(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1} \qquad (n \ge 1),$$

and put

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} M_k.$$

This is OEIS A187306 [O2]. Zhao [Z, Conjecture 1, p. 8] asks whether $(T_n^{1/n})_{n\ge 9}$ is log-concave. A prior public manuscript [P] claims strict log-concavity on that tail. The argument here instead uses a rational invariant interval and exact polynomial identities. The general passage from ratio log-concavity to root log-concavity is established prior methodology [CGW]; the elementary multiplicative identity is included to make the argument self-contained, without claiming that principle as new.

A positive sequence $(b_n)_{n\ge N}$ is *strictly log-concave* when $b_n^2 > b_{n-1}b_{n+1}$ at every center $n \ge N+1$. In particular, the statement that $(a_{n+1}/a_n)_{n\ge N}$ is strictly log-concave means

$$a_n a_{n+2}^3 > a_{n+1}^3 a_{n+3} \qquad (n \ge N).$$

These indexing conventions are important for sharpness.

Theorem 1 (original sequence). For all integers $n \ge 10$,

$$T_n T_{n+2}^3 > T_{n+1}^3 T_{n+3}. \tag{1}$$

For all integers $n \geq 6$,

$$T_n^{n^2-1} > T_{n-1}^{n(n+1)/2} T_{n+1}^{n(n-1)/2}. \tag{2}$$

Consequently the quotient tail starts at 10 and the root tail starts at 5. Both indices are optimal. In addition, for every $n \geq 12$,

$$T_n T_{n+2}^3 - T_{n+1}^3 T_{n+3} > \frac{800 T_n^4}{(n+3)^3}. \tag{3}$$

The constant 800 is a convenient certified constant, not an asserted optimum.

The next result covers arbitrary bounded recurrence errors. Define

$$L_n = 3 - \frac{9}{2(n+3)} + \frac{9}{16(n+3)^2}, \qquad U_n = L_n + \frac{2}{(n+3)^3}, \qquad E_n = \frac{1}{8(n+3)^2}. \tag{4}$$

Theorem 2 (bounded forcing). Let real numbers a_n, $n \geq 12$, satisfy

$$a_{12} > 1800, \qquad L_{12} < \frac{a_{13}}{a_{12}} < U_{12}, \tag{5}$$

and, for every integer $m \geq 14$,

$$(m+2)a_m = m(2a_{m-1} + 3a_{m-2}) + \varepsilon_m, \qquad |\varepsilon_m| \leq 2. \tag{6}$$

Then for all $n \geq 12$,

$$a_n > 8(n+3)^2, \qquad L_n < \frac{a_{n+1}}{a_n} < U_n, \tag{7}$$

and

$$a_n a_{n+2}^3 - a_{n+1}^3 a_{n+3} > \frac{800 a_n^4}{(n+3)^3} > 0. \tag{8}$$

There is no alternation or independence hypothesis on the errors.

Corollary (scaled forcing). Let $B \geq 0$. Replace the error bound in (6) by $|\varepsilon_m| \leq B$, and assume $a_{12} > 900B$ with the same strict quotient seed in (5). Then, for every $n \geq 12$,

$$a_n > 4B(n+3)^2, \qquad L_n < \frac{a_{n+1}}{a_n} < U_n,$$

and the quartic gap in (8) holds with the same constant 800. At $B = 0$, the first conclusion is $a_n > 0$.

For $B > 0$, apply Theorem 2 to $2a_n/B$. Its growth bound becomes $a_n > 4B(n+3)^2$, and division by the fourth power of the scale preserves (8). For $B = 0$, multiply the sequence by one sufficiently large positive constant so that its initial value exceeds 1800. Theorem 2 gives positivity and the quotient interval; cancellation of the same scale gives (8).

For the family extension, define the Riordan numbers by $R_0 = 1$, $R_1 = 0$, and

$$(n+1)R_n = (n-1)(2R_{n-1} + 3R_{n-2}) \qquad (n \geq 2).$$

This is the standard sequence [O1]. Put

$$A_n(s) = R_{n+1} + s(-1)^n = T_n + (s-1)(-1)^n, \qquad 0 \leq s \leq 1. \tag{9}$$

The equality in (9) is proved below. The endpoints are shifted Riordan numbers and the original alternating Motzkin sums.

Theorem 3 (uniform family). Every $A_n(s)$ with $n \geq 2$ is positive. Uniformly for $0 \leq s \leq 1$, the quotient inequality holds for $n \geq 10$, the root inequality holds at centers $n \geq 6$, and the quantitative inequality (8) holds for $n \geq 12$, with $a_n = A_n(s)$. The best tail indices for each individual parameter are completely classified in Section 6.

2. The inhomogeneous recurrence

From the definition,

$$M_n = T_n + T_{n-1} \qquad (n \geq 1).$$

For $n \geq 2$, let

$$D_n = (n+2)T_n - n(2T_{n-1} + 3T_{n-2}).$$

Substitution in the Motzkin recurrence at index $n-1$ gives, for $n \geq 3$,

$$\begin{aligned} D_n + D_{n-1} = (n+2)(T_n + T_{n-1}) \\ - (2n+1)(T_{n-1} + T_{n-2}) - 3(n-1)(T_{n-2} + T_{n-3}) = 0. \end{aligned}$$

The initial values give $D_2 = 2$. Therefore

$$(n+2)T_n = n(2T_{n-1} + 3T_{n-2}) + 2(-1)^n \qquad (n \geq 2). \tag{10}$$

Adding $(s-1)(-1)^n$ changes the forcing term to $2s(-1)^n$. At $s = 0$ the resulting homogeneous recurrence, and its first two values 0 and 1, identify it with R_{n+1} by induction. This proves (9) and

$$(n+2)A_n(s) = n(2A_{n-1}(s) + 3A_{n-2}(s)) + 2s(-1)^n. \tag{11}$$

The finite values used throughout are

n	0	1	2	3	4	5	6	7
T_n	1	0	2	2	7	14	37	90

n	8	9	10	11	12	13	14	15
T_n	233	602	1586	4212	11299	30536	83098	227474.

They follow directly from the defining recurrence. The independent audit also regenerates them from

$$M_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \frac{1}{j+1} \binom{2j}{j}.$$

This second construction is a computational cross-check, not an additional hypothesis of the proof.

3. Proof of the bounded-forcing theorem

3.1. Invariant interval and growth

For $n \geq 12$ one has

$$L_n - \frac{8}{3} = \frac{16n^2 - 120n - 477}{48(n+3)^2} > 0, \qquad 3 - U_n = \frac{72n^2 + 423n + 589}{16(n+3)^3} > 0. \tag{12}$$

For the first numerator, writing $n = 12 + t$ gives $16t^2 + 264t + 387$. Thus $8/3 < L_n < U_n < 3$.

Let

$$f_n(r, e) = \frac{n+1}{n+3}\left(2 + \frac{3}{r}\right) + \frac{2e}{n+3}.$$

For positive r, this is decreasing in r and increasing in e. These facts follow by subtraction of reciprocal fractions and require no differentiation.

The two assertions in (7) follow by joint induction. They hold at $n = 12$ by (5). Suppose they hold at $k-1$, where $k \geq 13$. Since $q_{k-1} = a_k/a_{k-1} > 8/3 > 2$,

$$a_k > 16(k+2)^2 > 8(k+3)^2;$$

the last inequality is $2(k+2)^2-(k+3)^2=k^2+2k-1>0$. The recurrence at $m=k+1$ gives

$$q_k=f_k(q_{k-1},e_k),\qquad e_k=\frac{\varepsilon_{k+1}}{2a_k},\qquad |e_k|<E_k.$$

Exact rational identities give

$$f_n(U_{n-1},-E_n)-L_n=\frac{304n^3+1687n^2+1599n-1790}{16(n+3)^3(48n^3+216n^2+297n+146)}, \tag{13}$$

$$U_n-f_n(L_{n-1},E_n)=\frac{112n^2-509n-1331}{16(n+3)^3(16n^2+40n+19)}. \tag{14}$$

All denominators are positive. At $n=13+t$, the two numerators become

$$304t^3+13543t^2+199589t+971988,\qquad 112t^2+2403t+10980,$$

so both are positive for $n\geq 13$. Monotonicity of f_n now proves $L_k<q_k<U_k$, completing the joint induction.

3.2. A local algebraic comparison

Fix $n\geq 12$. For independent real variables r,u,v, put

$$y=\frac{(n+2)(2r+3)+2u}{n+4},\quad z=\frac{(n+3)(2y+3r)+2v}{n+5},\quad F_n(r,u,v)=y^3-r^3z. \tag{15}$$

At

$$r=q_n,\qquad u=\frac{\varepsilon_{n+2}}{2a_n},\qquad v=\frac{\varepsilon_{n+3}}{2a_n},$$

the recurrence implies $y=a_{n+2}/a_n$ and $z=a_{n+3}/a_n$. Consequently

$$\frac{a_na_{n+2}^3-a_{n+1}^3a_{n+3}}{a_n^4}=F_n(r,u,v). \tag{16}$$

The invariant interval gives $8/3<r<3$ and $|u|,|v|<E_n\leq 1/1800<1/100$.

On the larger rectangular region

$$8/3\leq r\leq 3,\qquad |u|,|v|\leq 1/100, \tag{17}$$

one has $7\leq y\leq 9$ and $z\geq 19$. Indeed, the lower bound on $y-7$ is

$$\frac{200n-1703}{150(n+4)}>0,$$

the lower bound on $9-y$ is $899/[50(n+4)]>0$, and, using $y\geq 7$, the lower bound on $z-19$ is

$$\frac{150n-1451}{50(n+5)}>0.$$

Define the slopes

$$\alpha=\frac{2(n+2)}{n+4},\qquad \beta=\frac{(n+3)(7n+20)}{(n+4)(n+5)}.$$

Then $0<\alpha\leq 2$ and $\beta\geq 5$, the latter because

$$(n+3)(7n+20)-5(n+4)(n+5)=2n^2-4n-40>0.$$

For $r \le t$ with u, v fixed, write y_r, z_r and y_t, z_t for the corresponding values. Since $y_t - y_r = \alpha(t-r)$ and $z_t - z_r = \beta(t-r)$,

$$\begin{aligned} F_n(t,u,v) - F_n(r,u,v) = (t-r)\big[&\alpha(y_t^2 + y_t y_r + y_r^2) \\ &- (t^2 + tr + r^2)z_r - t^3\beta\big]. \end{aligned} \tag{18}$$

The bracket is at most

$$486 - 3(8/3)^2\,19 - (8/3)^3\,5 = -\frac{382}{27} < 0.$$

Thus F_n decreases in its first variable. The checker additionally verifies explicit nonnegative decompositions underlying this bound.

For $u_1 \le u_2$ with r, v fixed, put

$$\gamma = \frac{2}{n+4}, \qquad \eta = \frac{2(n+3)}{n+5} \le 2.$$

Direct factorization gives

$$F_n(r,u_2,v) - F_n(r,u_1,v) = (u_2-u_1)\gamma\big[y_2^2 + y_2 y_1 + y_1^2 - \eta r^3\big]. \tag{19}$$

The bracket is at least $147 - 54 = 93 > 0$, so F_n increases in its second variable. Finally,

$$F_n(r,u,v_2) - F_n(r,u,v_1) = -\frac{2r^3(v_2-v_1)}{n+5}, \tag{20}$$

so it decreases in its third variable. All intermediate comparison points remain in (17). Hence

$$F_n(q_n,u,v) \ge F_n(U_n, -E_n, E_n). \tag{21}$$

3.3. The corner certificate and quantitative gap

At the right-hand corner of (21), exact substitution gives

$$F_n(U_n,-E_n,E_n) = \frac{P(n)}{65536(n+3)^{11}(n+4)^3(n+5)}. \tag{22}$$

All coefficients of P are positive; they are listed in Appendix A. Here is a compact independently checkable form of the identity. Set $m = n+3$ and

$$\begin{aligned} &D = 16m^3, && V = 48m^3 - 72m^2 + 9m + 32, \\ &Y = (n+2)(2V+3D) - 4m, \\ &Z = m\big(2Y + 3V(n+4)\big) + 4m(n+4). \end{aligned}$$

At the corner, $r = V/D$, $y = Y/[D(n+4)]$, and $z = Z/[D(n+4)(n+5)]$. The polynomial identity

$$(n+5)DY^3 - (n+4)^2V^3Z = (n+3)P(n) \tag{23}$$

implies (22), since $D^4 = 65536(n+3)^{12}$.

For the explicit constant in (8), the second certificate is

$$P(n) - 800 \cdot 65536(n+3)^8(n+4)^3(n+5) = \sum_{j=0}^{12} c_j (n-12)^j, \tag{24}$$

where every c_j is strictly positive. Appendix A gives the exact coefficients. Equation (24) implies that (22) is strictly larger than $800/(n+3)^3$ for every $n \ge 12$. Combining (16), (21), and (22) proves Theorem 2. No limit, numerical optimization, or finite-range extrapolation is used.

4. Application to the parameter family

The two seeds are

$$A_{12}(s) = 11298 + s, \qquad A_{13}(s) = 30537 - s.$$

For $0 \le s \le t \le 1$,

$$\frac{30537 - s}{11298 + s} - \frac{30537 - t}{11298 + t} = \frac{41835(t - s)}{(11298 + s)(11298 + t)} \ge 0.$$

The endpoint comparisons are exact:

$$\frac{1081}{400} < \frac{30536}{11299}, \qquad \frac{30537}{11298} < \frac{145967}{54000}, \tag{25}$$

where $L_{12} = 1081/400$ and $U_{12} = 145967/54000$. The respective cross-product gaps are 181 and 137166. Also $11298 > 1800$. Equations (11) and (25) therefore satisfy Theorem 2 for every parameter in $[0, 1]$. This proves the quotient and quantitative claims for all $n \ge 12$.

The values $A_n(s)$ for $2 \le n \le 13$ are positive directly from the finite table in Section 2. Theorem 2 supplies positivity at all larger indices.

Let

$$Q_n(s) = A_n(s)A_{n+2}(s)^3 - A_{n+1}(s)^3 A_{n+3}(s). \tag{26}$$

For $2 \le n \le 11$, the quartic terms cancel and the resulting cubic coefficients are listed in Appendix B. The even-index cubics are strictly increasing on $[0, 1]$, and the odd-index cubics are strictly decreasing. A purely algebraic check is as follows. If $p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3$, then

$$\frac{p(t) - p(s)}{t - s} = c_1 + c_2(s + t) + c_3(s^2 + st + t^2) \quad (s < t).$$

For each even index, $c_3 > 0$, $c_2 < 0$, and $c_1 + 2c_2 > 0$. For each odd index, $c_3 < 0$ and $c_1 + 2\max(c_2, 0) < 0$. These facts prove the asserted strict monotonicity.

In particular,

$$Q_{10}(s) \ge Q_{10}(0) = 2279044915731 > 0,$$
$$Q_{11}(s) \ge Q_{11}(1) = 59137804243970 > 0.$$

This completes the uniform quotient statement for every $n \ge 10$.

5. Algebraic transfer to root log-concavity

For a positive sequence at the relevant indices, define

$$K_n = \frac{a_n^{n^2-1}}{a_{n-1}^{n(n+1)/2}\, a_{n+1}^{n(n-1)/2}} \qquad (n \ge 2).$$

Both half-exponents are integers. The root-log-concavity inequality at center n is equivalent to $K_n > 1$: raise the root inequality to the positive power $n(n^2 - 1)/2$.

Cancellation of monomials gives the exact identity

$$\frac{K_{n+1}}{K_n} = \left(\frac{a_{n-1} a_{n+1}^3}{a_n^3 a_{n+2}} \right)^{n(n+1)/2}. \tag{27}$$

For example, the exponents of $(a_{n-1}, a_n, a_{n+1}, a_{n+2})$ on both sides agree after expanding the four quadratic polynomials in n. The checker verifies these exponent identities in the doubled-exponent form, which avoids division by 2.

For $a_n = A_n(s)$, the quotient statement proved above makes the right-hand side of (27) larger than one whenever $n \geq 11$. It remains to check $K_n > 1$ for $6 \leq n \leq 11$. For a fixed center n, the root gap

$$H_n(s) = A_n(s)^{n^2-1} - A_{n-1}(s)^{n(n+1)/2} A_{n+1}(s)^{n(n-1)/2} \tag{28}$$

is strictly increasing in s when n is even, and strictly decreasing when n is odd. Indeed the center and its two neighbors vary in opposite directions, all bases are positive, and all exponents are positive. Hence the worst endpoints are 0 for even centers and 1 for odd centers. The six exact checks are

$$\begin{aligned} 36^{35} &> 15^{21}91^{15}, \\ 90^{48} &> 37^{28}233^{21}, \\ 232^{63} &> 91^{36}603^{28}, \\ 602^{80} &> 233^{45}1586^{36}, \\ 1585^{99} &> 603^{55}4213^{45}, \\ 4212^{120} &> 1586^{66}11299^{55}. \end{aligned} \tag{29}$$

These are integer comparisons, checked without floating point. Equation (27) propagates the last one to every later center. This proves the uniform root statement, and hence Theorem 3 apart from the sharp classification.

At $s = 1$, the immediate preceding obstructions are

$$Q_9(1) = -92071225688 < 0, \qquad 14^{24} < 7^{15}37^{10}. \tag{30}$$

The latter follows, after canceling 7^{15}, from

$$2^{24}7^9 = 677021181018112 < 4808584372417849 = 37^{10}.$$

Therefore the original quotient tail cannot start before 10 and the original root tail cannot start before 5. This proves all claims in Theorem 1.

A further quantitative consequence of (8) and $q_j < 3$ is available for any positive sequence satisfying Theorem 2. Whenever $n \geq 13$,

$$\frac{K_{n+1}}{K_n} > \left(1 + \frac{800}{729(n+2)^3}\right)^{n(n+1)/2}. \tag{31}$$

Indeed, the fraction inside (27) is $1 + Q_{n-1}/(a_n^3 a_{n+2})$, and $a_n^3 a_{n+2} < 729 a_{n-1}^4$. No claim is made that (31) alone supplies a positive base root gap for arbitrary initial seeds.

6. Exact sharp thresholds for every parameter

6.1. Quotient tails

For $j \in \{4, 5, 6, 7, 9\}$, let ρ_j be the unique zero of $Q_j(s)$ in $(0, 1)$. Strict monotonicity from Section 4 and the rational sign brackets in Appendix B establish existence and uniqueness. The order is

$$\rho_6 < \rho_4 < \rho_5 < \rho_7 < \rho_9. \tag{32}$$

Define $N_q(s)$ to be the least positive integer such that the quotient tail $(A_{n+1}(s)/A_n(s))_{n \geq N_q(s)}$ is defined, positive, and strictly log-concave. Then

Parameter interval	$N_q(s)$
$[0, \rho_6]$	7
$(\rho_6, \rho_4]$	5
(ρ_4, ρ_5)	3
$[\rho_5, \rho_7)$	6
$[\rho_7, \rho_9)$	8
$[\rho_9, 1]$	10

(33)

Here and below, equality at a boundary counts as failure of *strict* log-concavity.

To prove (33), note first that $Q_8(s) > 0$ throughout $[0,1]$, since Q_8 is increasing and $Q_8(0) = 68892649 > 0$. Also all Q_n with $n \geq 10$ are positive. The signs of Q_4, Q_5, Q_6, Q_7, Q_9 now follow from their unique roots and monotonicity. The last nonpositive indices in the six rows of (33) are respectively $6, 4, 2, 5, 7, 9$.

Only the third row requires an additional low-index observation. The isolating interval for ρ_5 lies below $6/25$. Direct substitution gives $Q_2(6/25) < 0$ and $Q_3(6/25) > 0$. Since Q_2 is increasing and Q_3 is decreasing, $Q_2(s) < 0 < Q_3(s)$ throughout (ρ_4, ρ_5). Thus the listed last nonpositive index really is 2. This proves every case and every endpoint in (33).

6.2. Root tails

For $j = 2, 3, 4, 5$, let τ_j be the unique zero in $(0,1)$ of $H_j(s)$ from (28). The four defining polynomials may be given without expansion:

$$\begin{aligned} H_2(s) &= (1+s)^3 - (1-s)^3(3-s), \\ H_3(s) &= (3-s)^8 - (1+s)^6(6+s)^3, \\ H_4(s) &= (6+s)^{15} - (3-s)^{10}(15-s)^6, \\ H_5(s) &= (15-s)^{24} - (6+s)^{15}(36+s)^{10}. \end{aligned} \tag{34}$$

The parity monotonicity argument applies also at these indices; at the endpoint $s = 1$, zero values such as $A_1(1) = 0$ do not disturb the polynomial sign argument. The brackets in Appendix B prove that each polynomial changes sign, and

$$\tau_4 < \tau_2 < \tau_3 < \tau_5. \tag{35}$$

There is a simple exact form for one boundary:

$$H_2(s) = -(s^2 - 6s + 1)(s^2 - s + 2), \qquad \tau_2 = 3 - 2\sqrt{2}. \tag{36}$$

Let $N_r(s)$ be the least positive integer such that the root tail $(A_n(s)^{1/n})_{n \geq N_r(s)}$ is positive and strictly log-concave. Then

Parameter interval	$N_r(s)$
$[0, \tau_4]$	4
$(\tau_4, \tau_2]$	2
(τ_2, τ_3)	1
$[\tau_3, \tau_5)$	3
$[\tau_5, 1]$	5

(37)

All centers at least 6 are already strictly positive. In the five rows, the last nonpositive centers among $2, 3, 4, 5$ are respectively $4, 2$, none, $3, 5$. Their signs follow directly from (34)–(35) and parity monotonicity. A last failing center j forces a root tail to start at least at j, not $j+1$, because its first internal center is one later than its starting index. In the middle row $s < 1$, so $A_1(s) = 1 - s > 0$, and all four small-center inequalities hold. Thus starting index 1 is valid and minimal. This proves (37).

The endpoint consequences are particularly simple: shifted Riordan numbers have the sharp quotient-tail start 7 and root-tail start 4; the alternating Motzkin sums have starts 10 and 5. The classification concerns $R_{n+1}^{1/n}$ at $s = 0$, not the differently indexed sequence $R_n^{1/n}$.

7. Verification scope and remaining work

The standard-library checker uses exact integer and rational polynomial arithmetic. It compares coefficients and shifted positivity certificates, regenerates finite data, and checks interval signs and base cases. The recorded `python -O -S` run did not depend on assertions or third-party packages.

A separate SymPy audit uses independently written expressions and finite Motzkin values from Catalan and binomial formulas. Deliberate certificate mutations test rejection. Finite parameter and index regressions are diagnostics and establish no infinite claim.

An exact checker is not a theorem prover. The written induction and comparisons link the identities to the universal conclusions.

The separate `formalization/` package proves Theorem 2 and the corrected scaled-forcing corollary for arbitrary real sequences and errors. It includes the simultaneous growth and quotient bounds and the quartic margin with constant 800 for every $n \geq 12$. The scaled growth is $a_n > 4B(n+3)^2$ for every $B \geq 0$, including positivity at $B = 0$. The stated seeds and recurrence onset are unchanged.

The package README gives the pinned Lean and Mathlib builds and the statement and axiom audits. Only `propext`, `Classical.choice`, and `Quot.sound` are permitted. Theorems 1 and 3, concrete-sequence identification, root transfer and both threshold classifications retain their conventional proofs and are outside this formalization.

The original conjecture has a prior public proof claim. The priority and significance of the extensions and parameter classifications remain unresolved. No certified first solution is claimed.

Research process and AI disclosure

This work is fully autonomous agentic research. Autonomous AI research agents produced the research, proofs, computations, formalization and drafts. GPT 6.1 Sol in Codex produced formal proofs and verification software for the accompanying package.

Opus 5.5 by Anthropic used with Claude Code revised this manuscript editorially on 1 November 2026: it added the abstract, moved this disclosure from the title page into its own section, converted the reference list into a bibliography with resolved citations, and placed the appendices after it. It did not change the mathematics.

Dev D. Goyal is responsible for publication. No new human or expert proof review is claimed, and the work has not been externally peer reviewed. Section 7 states the formal scope and the remaining work. All literature status statements refer to the sources inspected on 28 October 2026. A negative search result is not proof of novelty. Third-party full-text papers and manuscripts are not redistributed in this package.

References

[Z] Feng-Zhen Zhao, *The Log-Balancedness of the Sequence for the Alternating Sums of Motzkin Numbers*, INTEGERS **26** (2026), A55. Published 1 May 2026. DOI: 10.5281/zenodo.19949645. Journal-hosted paper. Conjecture 1 on page 8 is the original target; the paper's other results are not claimed here as new.

[CGW] William Y. C. Chen, Jeremy J. F. Guo, and Larry X. W. Wang, *Infinitely log-monotonic combinatorial sequences*, Advances in Applied Mathematics **52** (2014), 99–120. DOI: 10.1016/j.aam.2013.08.003. Publisher page. Ratio-to-root transfer is prior methodology.

[P] Public manuscript, *Motzkin alternating sums: strict log-concavity of the root sequence* (Chinese title; English description here), in the `xzwwwwww/math` repository. The relevant source was added on 17 October 2026 in commit `2e173134a5da24512ff9d2fbf1504b5634d29af8`. Pinned source. It claims strict root log-concavity at centers $n \geq 10$ by an integral method. This package records that public claim and does not independently certify every step of that manuscript or its publication status.

[O1] OEIS Foundation, A005043: Riordan numbers. Used to identify the standard homogeneous sequence; the necessary recurrence connection is proved in Section 2.

[O2] OEIS Foundation, A187306: alternating sum of Motzkin numbers. Sequence identification, not a source of the infinite inequalities.

Appendix A. Integer polynomial certificates

Write $P(n) = \sum_{j=0}^{12} p_j n^j$ and use c_j as in (24). Every entry below is positive. The machine-readable source is `proof/certificates.json`.

j	p_j	c_j
0	4660409556032	306994825721596657664
1	24132003073520	706171963822972745984
2	56628081108524	429272400422485948736
3	79753800966621	133590768426096292509
4	75163688028504	25822213571611442448
5	49975719381810	3377331981471101346
6	24049942325892	311625831704217912
7	8443039471929	20633351214169401
8	2146478817312	978753038000288
9	385445807712	32566142496352
10	46409331456	723458556672
11	3364180992	9653801984
12	111034368	58605568

Appendix B. Family cubics and exact root brackets

For $Q_n(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3$, the coefficients are:

n	c_0	c_1	c_2	c_3
2	-189	756	-36	43
3	2349	-9504	54	-102
4	-27189	134784	-234	250
5	479373	-2074888	-6	-628
6	-4865265	34034060	-2934	1608
7	160155377	-586941772	-9186	-4181
8	68892649	10545295603	-72309	11009
9	103947123741	-196017930893	-389241	-29295
10	2279044915731	3749897573668	-2548848	78655
11	132662633568165	-73524813169500	-15941880	-212815

The following intervals are exact rational isolating intervals, with common denominator 10^6. They are not floating-point root approximations used as proof. At both endpoints the checker evaluates the defining polynomial exactly, with the sign orientation determined by monotonicity.

Boundary	Lower numerator	Upper numerator
ρ_6	142954	142955
ρ_4	201778	201779

Boundary	Lower numerator	Upper numerator
ρ_5	231031	231032
ρ_7	272862	272863
ρ_9	530293	530294
τ_4	57244	57245
τ_2	171572	171573
τ_3	407203	407204
τ_5	530571	530572