Real positive-definite counterexamples to extended q-permanent monotonicity

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Abstract

An explicit real symmetric positive-definite matrix of order five has an inversion-weighted q-permanent that decreases between $q = 14$ and $q = 15$. Exactly nine permutations contribute nonzero terms. This refutes the real positive-definite monotonicity conjecture of da Fonseca [1, Conjecture 1], together with its later restatement [2, Conjecture 2]. A sparse family gives a counterexample at every prescribed real $q_0 > 1$. For each such parameter, a real symmetric positive-definite matrix has a strictly negative q-permanent derivative at q_0. The dimension may depend on q_0. Lean 4 formalizations establish the finite example, the theorem for every prescribed parameter, and the correlation and integer-entry consequences. The result does not address real-symmetric monotonicity on $[-1, 1]$.

1 The monotonicity conjecture

The ordinary permanent sums one entry product for each permutation. The q-permanent assigns an additional weight to each product. For an $n \times n$ matrix $A = (a_{ij})$, define

$$P_q(A) = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}, \qquad \mathrm{inv}(\sigma) = \#\{(i, j) : i < j,\ \sigma(i) > \sigma(j)\}.$$

The row and column order is fixed. A simultaneous permutation of rows and columns can change this polynomial. The convention is $0^0 = 1$.

Da Fonseca's 2010 paper uses $A > 0$ for a real symmetric positive-definite matrix. Under this convention, Conjecture 1 in Section 2 asserts strict monotonicity on an interval with left endpoint at or below -1 and right endpoint $+\infty$ [1]. Its later restatement uses a left endpoint strictly below -1 [2, Conjecture 2]. Either formulation implies

$$P_{q_1}(A) < P_{q_2}(A) \qquad (-1 < q_1 < q_2) \tag{1}$$

for every admissible non-diagonal real symmetric positive-definite A. The counterexample contradicts (1) at two positive integer parameters. It therefore refutes both formulations.

The Bapat–Lal monotonicity assertion is restricted to $[-1, 1]$. The October 2026 record of Kitamura's order-144 complex Hermitian counterexample explicitly leaves its real-symmetric case unsettled [4]. The present results concern $q > 1$.

Mitchell discusses a failure of the extended conjecture for positive semidefinite rank-one matrices at negative parameters [5]. That result does not supply the real positive-definite examples above one proved here. De Sá's earlier counterexamples concern incorrect tree expansion formulas [3]; the present proof uses a direct permutation sum and does not use those formulas.

Theorem 1 gives an exact five-dimensional counterexample. Theorem 3 gives, for every prescribed $r > 1$, a real symmetric positive-definite matrix whose q-permanent has a negative derivative at $q = r$; Corollaries 4 and 5 give correlation-matrix and integer-entry versions. Lean proves each of these statements (Section 5). An independent program also checks the finite examples by exhaustive enumeration.

2 A five-dimensional exact counterexample

Theorem 1. *The matrix*

$$A = \begin{pmatrix} 5 & 0 & 0 & 4 & 1 \\ 0 & 5 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & -1 & 0 & 5 & 0 \\ 1 & 4 & 0 & 0 & 5 \end{pmatrix}$$

is real symmetric, non-diagonal, and positive definite. Its q-permanent is

$$\begin{aligned} P_q(A) &= 625 + 25q^3 + 800q^5 + 25q^7 + 256q^8 - 32q^9 + q^{10} \\ &= 625 + 25q^3 + 800q^5 + 25q^7 + q^8(q-16)^2. \end{aligned} \tag{2}$$

It satisfies

$$P_{14}(A) = 8\,968\,822\,249 > 7\,441\,960\,000 = P_{15}(A),$$

with exact difference $1\,526\,862\,249$*. Consequently the real monotonicity conjecture described above is false.*

Proof. To prove positive definiteness, order the coordinates as $(1, 2, 4, 5, 3)$. The resulting matrix is the direct sum of [1] and

$$\begin{pmatrix} 5I_2 & R \\ R^T & 5I_2 \end{pmatrix}, \qquad R = \begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}, \qquad RR^T = 17I_2.$$

Its eigenvalues are 1 and $5 \pm \sqrt{17}$, with each of the latter occurring twice. All are positive. For the q-permanent calculation, retain the displayed order of the matrix in the theorem.

Every nonzero term fixes coordinate 3. Among the remaining coordinates, four transpositions correspond to nonzero off-diagonal entries. There are also two disjoint pairs of these transpositions and two oriented four-cycles. Together with the identity, these give exactly the nine terms in Table 1. Summing the table proves (2). Substitution of 14 and 15 yields the stated integer values, which contradict strict monotonicity. □

Permutation in one-line notation	Inversions	Entry product
$(1, 2, 3, 4, 5)$	0	625
$(1, 4, 3, 2, 5)$	3	25
$(1, 5, 3, 4, 2)$	5	400
$(4, 2, 3, 1, 5)$	5	400
$(4, 5, 3, 1, 2)$	8	256
$(4, 5, 3, 2, 1)$	9	-16
$(5, 2, 3, 4, 1)$	7	25
$(5, 4, 3, 1, 2)$	9	-16
$(5, 4, 3, 2, 1)$	10	1

Table 1: All nonzero terms for the fixed order in Theorem 1.

3 A sparse family

Let $m \geq 0$ be an integer and $n = m + 4$. The active coordinates are

$$i = 1, \qquad j = 2, \qquad k = m + 3, \qquad \ell = m + 4.$$

For real $a \geq 0$ and $d > \sqrt{a^2+1}$, define $A_m(a,d)$ as follows. It equals the identity on the m remaining coordinates. All entries between these coordinates and the active coordinates are zero. On the active coordinates, in increasing order, its principal submatrix is

$$\begin{pmatrix} d & 0 & a & 1 \\ 0 & d & -1 & a \\ a & -1 & d & 0 \\ 1 & a & 0 & d \end{pmatrix}. \tag{3}$$

It is positive definite, since the active block has eigenvalues $d \pm \sqrt{a^2+1}$, each twice. The intervening identity coordinates are part of the fixed ordering and must not be deleted when counting inversions.

Lemma 2. *For every $m \geq 0$,*

$$P_q(A_m(a,d)) = d^4 + d^2 q^{2m+1}(1 + 2a^2q^2 + q^4) + q^{4m+4}(a^2 - q)^2. \tag{4}$$

Proof. The isolated coordinates are fixed in every nonzero term. The following table uses cycle notation for the nine possible active permutations. Inversion counts use the full order $1, \ldots, m+4$.

Active permutation	Inversions	Entry product
id	0	d^4
$(j\ k)$	$2m+1$	d^2
$(i\ k)$, $(j\ \ell)$	$2m+3$	a^2d^2 each
$(i\ \ell)$	$2m+5$	d^2
$(i\ k)(j\ \ell)$	$4m+4$	a^4
$(i\ k\ j\ \ell)$, $(i\ \ell\ j\ k)$	$4m+5$	$-a^2$ each
$(i\ \ell)(j\ k)$	$4m+6$	1

For a transposition of positions $r < s$, the inversion count is $2(s-r)-1$. For each of the last three types, the four active coordinates contribute respectively 4, 5, and 6 inversions. Each intervening fixed coordinate contributes four additional inversions. These facts prove the stated counts. Summing gives

$$d^4 + d^2q^{2m+1} + 2a^2d^2q^{2m+3} + d^2q^{2m+5} + a^4q^{4m+4} - 2a^2q^{4m+5} + q^{4m+6},$$

which is (4). □

4 Failure at every prescribed parameter above one

The last term in (4) can have a negative derivative. The proof below chooses a and d in terms of m and a fixed $r > 1$. At $q = r$, this negative contribution grows faster in magnitude with m than the positive derivative of the middle term.

Theorem 3. *For every real $r > 1$, there are an integer $n \geq 5$ and a real symmetric positive-definite matrix A of order n such that*

$$\frac{d}{dq} P_q(A)\Big|_{q=r} < 0.$$

Thus no interval $[1, 1+\delta]$ with $\delta > 0$ can enjoy q-permanent monotonicity uniformly over all real symmetric positive-definite matrices of all orders.

Proof. Fix $r > 1$. Choose an integer $m \geq 1$ sufficiently large that

$$r^{2m+5} > (4m+4)(2m+5)\left(\frac{5r}{4}+2\right)\left(1+\frac{5}{2}r^3+r^4\right). \tag{5}$$

Such an integer exists. The left side grows exponentially in m. The right side is a quadratic polynomial in m times a constant that depends only on r.

Set

$$K = 4m+4, \qquad H = 2m+1, \qquad t = r\left(1+\frac{1}{K}\right), \qquad a = \sqrt{t}, \qquad d = \sqrt{t+2}.$$

Keep these parameters fixed when differentiating with respect to q. Then $d > \sqrt{a^2+1}$, so $A = A_m(a,d)$ is positive definite. By Lemma 2,

$$P_q(A) = d^4 + d^2 q^H (1 + 2tq^2 + q^4) + q^K (t-q)^2.$$

The derivative of the last summand at $q = r$ is exactly

$$\begin{aligned} Kr^{K-1}(t-r)^2 - 2r^K(t-r) &= Kr^{K-1}\frac{r^2}{K^2} - 2r^K\frac{r}{K} \\ &= -\frac{r^{K+1}}{K}. \end{aligned}$$

The derivative of the middle summand is

$$(t+2)r^{H-1}(H + 2t(H+2)r^2 + (H+4)r^4).$$

Since $K \geq 4$, the bound $t \leq 5r/4$ holds. Also $H, H+2 \leq H+4 = 2m+5$. Consequently this positive derivative is at most

$$\left(\frac{5r}{4}+2\right) r^{H-1}(2m+5)\left(1+\frac{5}{2}r^3+r^4\right).$$

Multiply (5) by r^{H-1}/K. The result shows that this upper bound is strictly smaller than r^{K+1}/K, since $K - H + 2 = 2m+5$. The total derivative is therefore negative.

Finally, given $\delta > 0$, apply the theorem with $r = 1+\delta/2$. A negative derivative of a polynomial gives a strictly decreasing pair of sufficiently nearby parameters contained in $[1, 1+\delta]$. □

Corollary 4. *The matrices in Theorems 1 and 3 can be replaced by real positive-definite correlation matrices, with the same strict monotonicity failures. At every prescribed $r > 1$, a counterexample can additionally be chosen with no zero entries.*

Proof. If $D = \operatorname{diag}(s_1, \ldots, s_n)$ with all $s_i > 0$, then term by term

$$P_q(DAD) = \left(\prod_{i=1}^{n} s_i^2\right) P_q(A).$$

Choose $s_i = a_{ii}^{-1/2}$ to normalize the diagonal. Positive definiteness is preserved, and the multiplicative factor is positive and independent of q.

For the final assertion, first replace a matrix from Theorem 3 by $A + \varepsilon \mathbf{1}\mathbf{1}^T$. This remains positive definite for every $\varepsilon > 0$. For sufficiently small positive ε, all entries are nonzero. The derivative remains negative because it is a polynomial in the entries and is therefore continuous. Then normalize the diagonal as above. □

Corollary 5. *For every real $r > 1$, a real symmetric positive-definite matrix with integer entries can be chosen such that $P_r'(A) < 0$.*

Proof. Positive definiteness and the strict inequality $P_r'(A) < 0$ are open conditions in the real vector space of symmetric matrices. Approximate the matrix of Theorem 3 by a sufficiently close rational symmetric matrix B preserving both conditions. Multiplication by a positive common denominator c gives an integer matrix, and $P_r'(cB) = c^n P_r'(B) < 0$. □

5 Formal verification and independent checks

The Lean formalization is in `lean/MathAndTheory/Mathematics/QPermanent/`. Its namespace is `MathAndTheory.QPermanent`. The table maps the paper's results to Lean declarations.

Paper statement	Lean declaration
Theorem 1, positivity	`counterexample_posDef`
Theorem 1, polynomial	`counterexample_polynomial`
Source conjecture is false	`not_daFonseca_extended_conjecture`
Lemma 2	`qPermanent_family_formula`
Theorem 3	`exists_posDef_negative_qPermanent_derivative_ge_five`
Correlation consequence	`exists_correlation_counterexample`
Dense correlation consequence	`exists_dense_correlation_negative_qPermanent_derivative`
Integer-entry consequence	`exists_integer_posDef_negative_qPermanent_derivative`

The formal definition sums over every permutation and counts ordinary inversions in the displayed coordinate order. The proof establishes the contribution of every intervening coordinate. Positive definiteness is proved by quadratic forms. The integer-entry proof uses rational approximations to the explicit family parameters, followed by denominator clearing. This gives the same existence conclusion as Corollary 5.

The independent program reconstructs the five-dimensional polynomial by enumerating all 120 permutations. It checks the positive rational LDL^T pivots $5, 5, 1, 8/5, 8/5$, both endpoint values, and $P'_{15}(A) = -1\,563\,030\,000$. For $m = 0, 1, 2, 3, 4$, it independently enumerates all permutations of the corresponding family matrix. It also checks the growth condition and negative derivative with exact rational arithmetic at $r = 101/100, 11/10, 3/2, 2, 15$. The respective choices of m are $925, 69, 12, 6, 1$. These computations check finitely many examples. Theorem 3 and its formalization prove the result for every r.

The repository pins Lean 4.35.0 and Mathlib commit `56a005855ea17e0b0958f73ec801ae140b0a1679`. See `lean/docs/mathematics/q-permanent.md` for statement alignment and `papers/mathematics/q-permanent/verification/` for commands, software versions, and recorded results. The Challenge file states the permutation definition directly and does not import the proof modules. Comparator checks that the Solution file proves the same statements.

6 Scope and prior work

The construction bounds the dimension of a counterexample. It does not determine the smallest possible dimension. The integer-entry and correlation conclusions are separate. They do not assert an integer correlation matrix. The results do not settle the real Bapat–Lal assertion on $[-1, 1]$.

No claim of first priority is made.

7 Research process and AI disclosure

This work is fully autonomous agentic research. Autonomous AI research agents produced the research, proofs, computations, formalization and drafts. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT, and GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities. This check is not expert validation or peer review.

Opus 5.5 by Anthropic used with Claude Code revised this manuscript editorially on 1 November 2026: it revised the abstract and this disclosure and added a summary of the results to Section 1. It did not change the mathematics.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed. Section 5 states the formal scope, and Section 6 states the limits of the result.

References

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[4] Kenta Kitamura. Lean 4 disproof of q-permanent monotonicity (order 144), 2026. URL `https://github.com/KitaKen1/bapat-lal-q-permanent-lean/tree/4200da4fc1a132d69c23fb877795b5b72089544b`. Public formalization; immutable source revision.

[5] L. Mitchell. A note on Bapat's q-permanent conjecture. *Operators and Matrices*, 14(4): 915–919, 2020. doi: 10.7153/oam-2020-14-56.