Order-polynomial positivity for width-two posets with a chain of at most five elements

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Abstract

For a finite poset P let $\Omega_P(t)$ be its order polynomial, which counts weakly order-preserving maps $P \to \{1, \dots, m\}$ at $t = m$. Ferroni, Morales and Panova [2] conjectured that $|P|!\,\Omega_P(t)$ has nonnegative coefficients in the monomial basis whenever P has width at most two, and reported computer checks when both chains of a two-chain cover have at most five elements. We prove the conjecture for every finite poset that is the union of two chains A and B with $|A| \le 5$, the chain B being arbitrarily long and the comparabilities between A and B arbitrary. For $|A|, |B| \ge 1$ we also prove $[t]\,\Omega_P(t) \ge (|A|-1)!\,(|B|-1)!\,m/|P|!$, where m is the number of elements of B comparable with some element of A, so that the linear coefficient is positive exactly when the comparability graph is connected. Via the descent–rook correspondence of Alexandersson and Jal [1], the normalized linear coefficient is, on each of the 64 chambers of endpoint configurations, a polynomial of degree at most $|A|$ in the gaps between endpoints, independent of $|B|$. Exact certificates (sums of 1,072 rational squares plus nonnegative remainders, matched with an independent order-polynomial evaluation at 194,112 determining points) prove these polynomials nonnegative, and the known ideal–filter reduction of Ferroni, Morales and Panova lifts the linear coefficient to all coefficients. Lean 4 checks the nonnegativity certificates of the chamber polynomials for $|A| \le 4$; the rest is a conventional proof with certificates checked by an independent program.

1 Introduction and results

Let P be a finite poset with n elements. For a positive integer m let $\Omega_P(m)$ be the number of maps $f : P \to \{1, \dots, m\}$ with $f(x) \le f(y)$ whenever $x < y$. It is a polynomial in m of degree n, the *order polynomial*, and $\Omega_P(t+1)$ is the Ehrhart polynomial of the order polytope of P [6]. Writing $\Omega_P(t) = \sum_k e_k \binom{t}{k}$, where e_k counts chains $\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_k = P$ of order ideals, shows that $n!\,\Omega_P(t) \in \mathbb{Z}[t]$. Nonnegativity of the coefficients of $\Omega_P(t)$ implies that of $\Omega_P(t+1)$, hence Ehrhart positivity of the order polytope, but it fails in general: the star with one minimum and four maximal elements has $\Omega(t) = \frac{t^5}{5} + \frac{t^4}{2} + \frac{t^3}{3} - \frac{t}{30}$.

The conjecture. Ferroni, Morales and Panova [2, Section 7.1, Conjecture 7.1] conjectured that $n!\,\Omega_P(t)$ has nonnegative coefficients in the monomial basis for every poset P of width at most two. By Dilworth's theorem these are exactly the posets covered by two chains. They reported computer verification when both chains have at most five elements, proved positivity for the cell posets of skew shapes (a different construction of a poset from a shape), and showed that for a family closed under order ideals and their complementary filters positivity of all coefficients follows from positivity of the linear coefficient [2, Proposition 3.3, Theorem 3.4].

Theorem 1. *Let P be a finite poset that is the disjoint union of two chains A and B with $|A| \le 5$. There is no condition on $|B|$ or on the comparabilities between A and B. Then every coefficient of $|P|!\,\Omega_P(t)$ in the monomial basis is a nonnegative integer.*

Theorem 2. *In Theorem 1 let $a = |A| \geq 1$, $b = |B| \geq 1$ and let $m_A(B)$ be the number of elements of B comparable with at least one element of A. Then*

$$[t]\,\Omega_P(t) \;\geq\; \frac{(a-1)!\,(b-1)!}{(a+b)!}\, m_A(B).$$

Corollary 3. *For a nonempty poset P as in Theorem 1, the linear coefficient of $\Omega_P(t)$ is positive if the comparability graph of P is connected and zero otherwise.*

The empty poset ($\Omega = 1$) and the case of an empty chain are included. The hypothesis is a partition into two chains with one of them short; the existence of a long chain whose complement has at most five elements is weaker, since the complement need not be a chain. The case $a = 1$ is elementary (Example 8). Five is the limit of the certificates we have, not a proven threshold.

Method. Section 2 encodes the comparabilities between the chains by two monotone boundary sequences. Using the correspondence between descents of linear extensions and non-nesting rook placements due to Alexandersson and Jal [1], Section 3 writes $n!\,[t]\Omega_P(t)/(b-1)!$ as an alternating combination of rook numbers. Section 4 splits the boundary configurations into C_a chambers, C_a the Catalan number, and shows that on each chamber this functional is a polynomial of degree at most a in the $2a+1$ gaps between consecutive endpoints, with no dependence on b beyond the gaps. Finitely many exact evaluations therefore identify it, and Section 5 certifies each chamber polynomial nonnegative on the orthant. For $a \leq 3$ all its coefficients are positive; for $a = 4$, ten of the fourteen chamber polynomials have negative coefficients and are written as sums of rational squares plus a remainder with nonnegative coefficients. Section 6 applies the ideal–filter reduction.

Related work. Section 9 compares the result with two public partial results on the same conjecture: positivity for ordinal sums of one- and two-element antichains, and a coefficient positivity theorem for lattice-point counts whose order-polynomial factors have a single cross-link. Neither class contains the class of Theorem 1, nor is contained in it.

2 Boundary encoding

Write $A_1 < \cdots < A_a$ and $B_1 < \cdots < B_b$. For $1 \leq i \leq a$ let L_i be the number of elements of B below A_i and U_i be b minus the number of elements of B above A_i. By transitivity within the chains,

$$0 \leq L_1 \leq \cdots \leq L_a \leq b, \qquad 0 \leq U_1 \leq \cdots \leq U_a \leq b, \qquad L_i \leq U_i, \tag{1}$$

and the cross relation is

$$B_j < A_i \iff j \leq L_i, \qquad A_i < B_j \iff j > U_i, \qquad A_i \parallel B_j \iff L_i < j \leq U_i. \tag{2}$$

Conversely, any data satisfying (1), with (2) and the two chain orders, define a partial order. Antisymmetry holds since $j \leq L_i$ and $j > U_i$ are incompatible. For transitivity through the other chain: $A_i < B_j < A_{i'}$ gives $U_i < j \leq L_{i'} \leq U_{i'}$, hence $i < i'$ by monotonicity of U; $B_j < A_i < B_{j'}$ gives $j \leq L_i \leq U_i < j'$; the remaining patterns combine one cross relation with a chain relation and follow from monotonicity. Every pair (L, U) satisfying (1) therefore arises, and the incomparable pairs form the board of cells (i, j) with $L_i < j \leq U_i$.

3 The linear coefficient as a rook functional

A *non-nesting rook placement* of size k is a set of k incomparable pairs $(A_{i_1}, B_{j_1}), \ldots, (A_{i_k}, B_{j_k})$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$; let r_k be their number, $r_0 = 1$.

Lemma 4 (Alexandersson–Jal)**.** *Label P naturally by a linear extension that, whenever both are available, takes an element of A before an element of B. Then the number of linear extensions of P with k descents is r_k.*

Proof. This is the correspondence of [1, Theorem 5.4, Lemma 5.6, Corollary 5.8]; we give the direct argument covering empty rows, columns and coincident endpoints. If $A_i \parallel B_j$ then $L_i < j$, and by induction on i every element below A_i precedes B_j in the labelling, so A_i is available before B_j is chosen and receives the smaller label. A descent of a linear extension is a pair of adjacent incomparable elements in decreasing label order, hence a pair B_jA_i with $A_i \parallel B_j$, and the recorded pairs of an extension have increasing indices in both chains. Conversely, given a non-nesting placement, the prefixes $\{A_1,\dots,A_{i_s-1}\}\cup\{B_1,\dots,B_{j_s-1}\}$ and $\{A_1,\dots,A_{i_s}\}\cup\{B_1,\dots,B_{j_s}\}$ are order ideals by $L_{i_s} < j_s \le U_{i_s}$ and (1); filling the elements between consecutive prescribed pairs in increasing label order gives the unique linear extension whose descents are exactly the prescribed pairs. □

By the standard descent expansion [6, Section 3.15], for a natural labelling,

$$\Omega_P(t) = \sum_{k=0}^{a} r_k \binom{t+n-1-k}{n}, \qquad n = a+b. \tag{3}$$

Assume $a, b \ge 1$. For $0 \le k \le n-1$ the binomial $\binom{t+n-1-k}{n}$ is $(t+n-1-k)\cdots(t-k)/n!$, whose linear coefficient is $(-1)^k k!\,(n-1-k)!/n!$. Since $k \le a \le n-1$,

$$F := \frac{n!\,[t]\,\Omega_P(t)}{(b-1)!} = \sum_{k=0}^{a} (-1)^k\, k!\, b^{\overline{a-k}}\, r_k, \qquad b^{\overline{d}} = b(b+1)\cdots(b+d-1). \tag{4}$$

This is the linear coefficient of the unshifted order polynomial, not of the Ehrhart polynomial $\Omega_P(t+1)$.

4 Chambers and gap polynomials

Merge the lists $L_1,\dots,L_a$ and $U_1,\dots,U_a$ into one weakly increasing list, placing L-endpoints before U-endpoints at ties and keeping the order within each list. Recording the endpoint types gives a word w with a letters L and a letters U; since $L_i \le U_i$, every prefix has at least as many L's as U's, so w is a Dyck word. There are $C_a = \frac{1}{a+1}\binom{2a}{a}$ of them: $1, 2, 5, 14, 42$ for $a = 1,\dots,5$, in total 64 chambers. Let x_0 be the first endpoint, x_j the distance from the j-th to the $(j+1)$-st endpoint for $1 \le j < 2a$, and $x_{2a} = b$ minus the last endpoint. Then $x \in \mathbb{Z}_{\ge 0}^{2a+1}$, $b = x_0 + \cdots + x_{2a}$, and conversely every pair (w, x) gives boundary data satisfying (1), hence a poset. Ties are zero gaps; a configuration on a chamber boundary may have several encodings, which is harmless.

If after j endpoints o letters L and c letters U have occurred, every element $B_{j'}$ in the j-th gap segment is incomparable exactly with $A_{c+1},\dots,A_o$, by (2). Write $h_j = o - c$ for the number of these rows; $h_0 = h_{2a} = 0$.

Lemma 5. *For fixed (a, w) there is a polynomial $F_{a,w} \in \mathbb{Q}[x_0,\dots,x_{2a}]$ of total degree at most a, with $F_{a,w}(0) = 0$, such that F in (4) equals $F_{a,w}(x)$ for every poset encoded by (w, x) with $b \ge 1$.*

Proof. A placement of size k selects k rows and assigns them, in increasing order, to weakly increasing gap segments in which they are active; if segment j receives d_j rows, their columns can be chosen in $\binom{x_j}{d_j}$ ways, and the columns are then increasing. Hence $r_k = \sum \prod_j \binom{x_j}{d_j}$ over the finitely many such assignments, a polynomial of degree at most k. Since $b = \sum x_j$, each term of (4) has degree at most a. At $x = 0$ every r_k with $k \ge 1$ vanishes and $0^{\overline{a}} = 0$. □

Lemma 6. *A polynomial of total degree at most a in d variables that vanishes on $S_{d,a} = \{x \in \mathbb{Z}_{\ge 0}^d : x_0 + \cdots + x_{d-1} \le a\}$ is zero.*

Proof. The products $\prod_i \binom{x_i}{\alpha_i}$ with $|\alpha| \le a$ form a basis. Evaluating at $x = \beta \in S_{d,a}$ in order of increasing $|\beta|$, only $\alpha \le \beta$ contribute, the coefficients with $|\alpha| < |\beta|$ are already zero, and among $|\alpha| = |\beta|$ only $\alpha = \beta$ survives, with value one. □

Thus $F_{a,w}$ is determined by its values on $S_{2a+1,a}$, which has $\binom{3a+1}{a}$ points: $4, 21, 120, 715, 4368$ for $a = 1, \dots, 5$.

5 Certificates

For each of the 64 chambers the certificate supplies $F_{a,w}$ explicitly by its monomial coefficients, and a proof that it is nonnegative on $\mathbb{R}^{2a+1}_{\ge 0}$:

- for $a \le 3$, all coefficients of $F_{a,w}$ are positive integers;
- for $a = 4, 5$, an identity

$$F_{a,w}(x) = \sum_s q_s \, x^{\mu_s} \left(\rho_s x^{u_s} - x^{v_s}\right)^2 + \sum_e c_e \, x^e \tag{5}$$

with rational $q_s \ge 0$, $c_e \ge 0$, rational ρ_s and exponent vectors $\mu_s, u_s, v_s, e \in \mathbb{Z}^{2a+1}_{\ge 0}$; every term is nonnegative on the orthant.

a	chambers	points per chamber	determining points	square terms
1	1	4	4	0
2	2	21	42	0
3	5	120	600	0
4	14	715	10,010	101
5	42	4,368	183,456	971
total	64		194,112	1,072

Table 1: Size of the certificate.

The checker `certificate/check_certificates.py`, which uses only the Python standard library and exact rational arithmetic, verifies:

1. the chamber list for each $a \le 5$ is exactly the set of Dyck words;
2. each $F_{a,w}$ has $2a + 1$ variables, total degree at most a and zero constant term;
3. positivity of the coefficients for $a \le 3$; for $a = 4, 5$ nonnegativity of every q_s and c_e and exact coefficientwise equality in (5);
4. at every point $x \in S_{2a+1,a}$, equality of $F_{a,w}(x)$ with the value of (4) for the poset encoded by (w, x), computed from scratch: $\Omega_P(m)$ for $m = 0, \dots, n$ by counting multichains of order ideals, which are pairs of chain prefixes, and the linear coefficient from finite differences; this evaluation does not use rook numbers;
5. the degree-one remainder coefficients needed in Section 7.

As a sanity check of Lemma 5, the checker also compares $F_{a,w}$ with (4) at three random gap vectors with $|x| > a$ in every chamber, and it rejects corrupted certificates, among them a polynomial and remainder shifted by the same monomial, which still satisfy (5) and are caught only by the point identities. The certificates and the identities were found with the help of a linear-programming search; the search is not part of the proof, and the accepted weights are

exact rationals. The original research package contained two further independent verifications of the same identities, which we reran.

By Lemmas 5 and 6 the identities hold as polynomial identities, so $F_{a,w}(x) \geq 0$ for every $x \geq 0$, and by (4) $[t]\,\Omega_P(t) \geq 0$ for every P in Theorem 1 with $a, b \geq 1$.

Remark 7. For $a = 4$ the coefficientwise argument is not available: ten of the fourteen polynomials $F_{4,w}$ have a negative coefficient. A negative coefficient of a gap polynomial is not a negative value of an order polynomial. For $a = 6$ two limited searches for certificates of the form (5) failed; this says nothing about the truth of the conjecture there.

6 Proof of Theorem 1

The class $\mathcal{S}_5$ of posets admitting a partition into two chains, one of size at most five, is closed under order ideals and their complementary filters: intersect the two chains with the ideal or filter. For a nonempty poset P write $\Omega_P(t) = \sum_{k\geq 1} c_k(P)t^k$; $\Omega_P(0) = 0$. Splitting a map to $\{1, \ldots, x + y\}$ at the largest value at most x gives $\Omega_P(x + y) = \sum_I \Omega_I(x)\,\Omega_{P\setminus I}(y)$ over order ideals I; putting $x = y = t$ and comparing coefficients,

$$(2^k - 2)\,c_k(P) = \sum_{\emptyset \neq I \neq P} \sum_{i=1}^{k-1} c_i(I)\,c_{k-i}(P \setminus I) \qquad (k \geq 2). \tag{6}$$

This is the argument of [2, Proposition 3.3, Theorem 3.4]. By induction on $|P|$ the right-hand side is nonnegative for $P \in \mathcal{S}_5$, and $c_1(P) \geq 0$ by Section 5 when both chains are nonempty. If a chain is empty, P is a chain and $\Omega_P(t) = t(t + 1)\cdots(t + n - 1)/n!$. Integrality was noted in the introduction. □

7 The linear lower bound

The checker also verifies that in every chamber the degree-one part of the nonnegative part of the certificate (the polynomial itself for $a \leq 3$, the remainder in (5) for $a = 4, 5$) is exactly

$$(a - 1)! \sum_{j:\, h_j < a} x_j.$$

All other terms are nonnegative on the orthant, so $F_{a,w}(x) \geq (a - 1)! \sum_{h_j<a} x_j$. An element of B in gap segment j is incomparable with exactly h_j elements of A, so it is comparable with some element of A if and only if $h_j < a$, and the sum equals $m_A(B)$. Multiplying by $(b - 1)!/n!$ proves Theorem 2.

For Corollary 3: if $a = 0$ or $b = 0$ the poset is a nonempty chain, which is connected and has linear coefficient $1/n$. Otherwise, if some element of B is comparable with some element of A then $m_A(B) \geq 1$, the comparability graph is connected because each chain is, and the linear coefficient is positive. If there is no such comparability, P is the disjoint union of two nonempty chains, Ω_P is the product of their order polynomials, each without constant term, and the linear coefficient is zero. □

Example 8. For $a = 1$ with boundaries $L \leq U$ there are $U - L$ incomparable elements, $r_1 = U - L$ and $F = b - (U - L) = m_A(B)$, so Theorem 2 is an equality: $[t]\,\Omega_P(t) = m_A(B)/(b(b + 1))$. For $b = 8$, $L = 2$, $U = 6$ it is $1/18$.

Example 9. For $a = 5$, $b = 20$, $L = (0, 2, 5, 5, 11)$ and $U = (4, 8, 12, 17, 20)$, every element of B is comparable with some element of A, and Theorem 2 gives $[t]\,\Omega_P(t) \geq 4!\,19!\,20/25! = 1/265650$; the exact value is $7123/531300$. This poset is outside the range of the computations in [2]. For $L = (0, \ldots, 0)$ and $U = (20, \ldots, 20)$ there are no cross-comparabilities and $\Omega_P(t) = \binom{t+4}{5}\binom{t+19}{20}$ has zero linear coefficient.

8 Formal verification

The Lean 4 project (Lean 4.35.0, Mathlib `56a005855ea17e0b0958f73ec801ae140b0a1679`) checks the chamber certificates for $a \le 4$. A chamber is a record of the word w, a scale D and integer data: the polynomial $D\,F_{a,w}$, the degree-one part and the rest of D times the nonnegative remainder, and square triples (p, q, r) standing for $p\,x^{\mu+2u} - q\,x^{\mu+u+v} + r\,x^{\mu+2v}$. The file `ShortChain/Data.lean` is generated from the certificate files by `scripts/generate_lean_data.py`, which also checks that it matches them. The Boolean test `Chamber.valid` is evaluated by the kernel (`decide +kernel`) for all 22 chambers: the remainder and the expanded squares add up to the polynomial, the remainder coefficients and p, r are nonnegative and $q^2 \le 4pr$, and the degree-one part is $D\,(a-1)!$ exactly on the segments with $h_j < a$. The theorem `ShortChain.certified_chambers` states that for every chamber in the list and every gap vector x with nonnegative entries in any linearly ordered field,

$$0 \le F_{a,w}(x) \quad \text{and} \quad (a-1)! \sum_{h_j < a} x_j \le F_{a,w}(x),$$

and `ShortChain.chambers_complete` that for $a = 1, \dots, 4$ the chamber words are exactly the Dyck words. No `native_decide` is used, and a module-level audit shows that every declaration depends only on `propext`, `Classical.choice` and `Quot.sound`.

Not formalized are the identification of $F_{a,w}$ with the linear coefficient of the order polynomial (Lemmas 4 and 5, the descent expansion (3), the determining-set Lemma 6 and the point identities), the chambers with $a = 5$, whose 177,176 polynomial terms are checked only by the Python program, and the reduction of Section 6.

9 Related work and scope

The conjecture, the reduction to the linear coefficient and the computations for both chains of size at most five are from [2]; the descent–rook correspondence is from [1], and the descent expansion is classical [6]. Liu, Xin and Zhang [5] proved Ehrhart positivity of order polytopes of dimension at most 13, which concerns $\Omega_P(t+1)$ and bounded $|P|$. Jochemko and Menon [3] showed that for an ideal–filter closed family with nonnegative linear coefficients, marked order polynomials have nonnegative coefficients in the marking gaps; Theorem 1 supplies that hypothesis for $\mathcal{S}_5$. Kahane [4] interprets the coefficients for fences.

Two public partial results on Conjecture 7.1 of [2] were found while preparing this note. The repository 10m-u/open-problems-catalog (commit of 14 October 2026) proves nonnegativity for all ordinal sums of antichains of size one or two, by a root-location argument. Such a poset lies in $\mathcal{S}_5$ exactly when it has at most five two-element layers, while $\mathcal{S}_5$ contains posets outside the closure of these layers under ordinal and disjoint sums, the smallest being the four-element poset with relations $a < b$, $c < b$, $c < d$, whose comparability and incomparability graphs are both connected. The repository chris-hadad/lr-stretching-results (commit of 29 October 2026) proves coefficient positivity for a lattice-point count whose order-polynomial factors are two-chain posets with a single cross-link; those factors may have both chains longer than five, but their cross-relations are far more restricted than in $\mathcal{S}_5$. Neither result implies Theorem 1, and Theorem 1 implies neither. The present theorem is not the first all-size partial result on the conjecture. Searches in October 2026 found no statement of Theorem 1 or 2; coverage of recent and unpublished work is incomplete and no priority is claimed.

The unrestricted conjecture remains open. The next case of the method is $a = 6$, with 132 chambers of polynomials in 13 gaps.

10 Research process and AI disclosure

This is fully autonomous agentic research. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed, no expert validation is claimed, and the novelty assessment is provisional.

The theorem, the chamber reduction, the certificates and their verifiers were produced by autonomous AI research agents in a GPT-6 Astra research workflow, according to its records. The workflow's internal reviews, including a separate referee reconstruction of the rook polynomials, were AI-assisted reviews of the proof and not peer review. Its own late audit found the two public partial results discussed in Section 9.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It checked the proof, wrote an independent certificate checker with its own order-polynomial evaluation, reran the original verifications, compared the statement with the cited sources and the two public partial results, wrote and ran the Lean formalization described in Section 8, and wrote this note. It also made the editorial revision of 1 November 2026, which did not change the mathematics. AI checks are not human peer review. Section 8 states the formal scope and Section 9 the remaining limits.

References

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