Graphs with five harmonic boundary vertices, subcubic graphs with six, and minimum-boundary two-trees

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Abstract

We give a computer-assisted classification of all finite simple connected graphs with exactly five Steinerberger (harmonic) boundary vertices. The key theorem is that every two-connected graph with at most five such vertices has diameter at most two. An all-diameter reduction uses the three classical cones of five-point metrics and preserves boundary-radius-two profile configurations while capping their parameters. An exact local certificate leaves only fifteen bounded representatives, excluded by hand. Independent programs and explicit contradiction trees certify the finite step. The resulting classification consists of exactly 227 reduced kernels, with arbitrary subdivisions of bridges; their order is at most fourteen, sharply. The six-boundary sequel is completely solved for subcubic graphs: there are twelve two-connected cores and exactly 201 reduced connected kernels, of order at most eighteen, with two equality kernels. We classify every equality case in the sharp two-tree bound: exactly three at each even order at least eight, and an explicit period-eight quartic quasipolynomial at odd orders, asymptotic to $n^4/192$. A canonical core decomposition gives the exact rational generating function. More generally, every fixed excess above the two-tree bound admits a finite grammar with a polynomial state bound and an effectively computable rational unlabelled generating function. For every boundary size, bipartite subcubic graphs have boundary exactly at their leaves and cyclic vertices; boundary six gives ten kernels. A general rooted metric theorem has sharp bounds $2R + 1$ and $4R + 2$, while an infinite family proves that the stated radius-two tests alone cannot certify global realizability at any boundary size at least six. The general metric theorem and full bipartite subcubic boundary theorem are proved in Lean. Exact certificates, formalization scope, and remaining six-boundary limitations are specified explicitly.

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1 Setting, prior work, and status

All graphs in this note are finite, simple, undirected, connected, and have at least two vertices. Distances are unweighted shortest-path distances. Put

$$\beta_G(x,y) = \sum_{z \in N_G(x)} (d_G(x,y) - d_G(z,y)), \qquad \beta_G(x) = \max_{y \in V(G)} \beta_G(x,y).$$

The boundary in the question is exactly $\partial G = \{x : \beta_G(x) > 0\}$. Write $b(G) = |\partial G|$.

Steinerberger introduced this boundary [2]. Chiem et al. introduced the stability number, proved the bridge rule and the classification through boundary four, and posed boundary five as Problem 6.1 [1]. Their Theorem 5.2 describes the boundary at diameter at most two. These ingredients are credited prior results, not contributions of this note. The 2025 potential-theory paper [3] does not settle the classification.

The main new result is the unrestricted classification. Its finite step is supported by a uniform parameter reduction, not by extrapolation from a bounded graph census. The three-cone description of five-point metrics is classical [5, 6]; the local capping argument and the harmonic infeasibility certificate are proved here. The sharp two-tree theorem is strengthened here to complete equality grammars, a parity-dependent counting theorem, and effective grammars and rational unlabelled generating functions at every fixed excess.

A targeted literature search in November 2026 found no earlier classification of graphs with five harmonic boundary vertices and no earlier statement of the sharp two-tree bound. Such a search cannot establish priority. How the results were obtained and checked is described in Section 20.

Theorem 1.1 (Main classification)**.** *Every two-connected graph with at most five harmonic boundary vertices has diameter at most two. Exactly eighteen two-connected graphs have boundary five: eight of order five and ten of order six. Every connected graph with boundary five is obtained by independently subdividing bridges of one of exactly 227 reduced kernels. The largest reduced order is fourteen, with a unique equality kernel.*

The core theorem is proved in Theorem 5.1; the explicit block palette and complete assembly are given in Lemma 8.1 and Theorem 8.3. A reduced kernel suppresses degree-two vertices only when they lie on no cycle. Thus the theorem describes all graph orders and all diameters.

Proof status. The graph and metric reductions are proved in prose. The finite local step has independently generated exact certificates and a separate verifier. The 227-kernel count has a complete closure certificate. The whole classification is not claimed to be formalized end to end in Lean.

2 A flux theorem and a Lean-verified six-boundary obstruction

For any vertex set W and target t, internal edges cancel to give

$$\sum_{x \in W} \beta(x,t) = \sum_{xy \in E:\, x \in W,\, y \notin W} (d(x,t) - d(y,t)). \tag{1}$$

Strictly positive outgoing flux therefore certifies a boundary vertex in W. Say a root u has the *unique first-step property* if every $x \neq u$ has exactly one neighbor v_i of u with $d(v_i,x) = d(u,x) - 1$. Put $f(x) = d(u,x)$ and let S_i be the sector of vertices with that first neighbor. The sectors partition $V \setminus \{u\}$.

Theorem 2.1 (Sector lower bound)**.** *Let u have the unique first-step property, and let J be the set of indices i for which there is a nonadjacent root neighbor v_j and an $x \in S_i$ with $d(v_j,x) = f(x)$.*

Then

$$b(G) \geq \deg(u) + |J| + \mathbf{1}_{\{u \in \partial G\}}.$$

If u is not a cut vertex and $G[N(u)]$ has $c \geq 2$ components, then

$$b(G) \geq \deg(u) + c + \mathbf{1}_{\{u \in \partial G\}}.$$

In particular, if u is not a cut vertex, independent neighbors and $\deg(u) \geq 2$ give $b(G) \geq 2\deg(u) + \mathbf{1}_{\{u \in \partial G\}}$.

Proof. An edge between distinct sectors preserves f: a rise of one would produce a shortest path with a second first neighbor. Consequently the outgoing f-flux of S_i is exactly one, from the edge $v_i u$. Each sector contains a boundary vertex.

For $i \in J$ choose j as in the statement and partition

$$P = \{x \in S_i : d(v_j, x) = f(x)\}, \qquad T = S_i \setminus P.$$

The set P is nonempty and $v_i \in T$, because v_i, v_j are nonadjacent. For an edge xy from P to T, distance Lipschitzness gives $f(x) \geq f(y)$, since $d(v_j, y) = f(y) + 1$. Edges from P to other sectors preserve f, and none goes to u. A vertex of P minimizing f has a geodesic predecessor outside P, giving a strictly positive contribution. Thus P has positive f-flux.

For T use target v_j. The edge $v_i u$ contributes $2 - 1 = 1$. An edge xy from T to P contributes $f(x) + 1 - f(y) \geq 0$. An edge leaving S_i preserves f and cannot enter S_j, since its distances to v_j would differ by two. In every other sector its endpoint has distance at most $f(x) + 1$ from v_j, giving a nonnegative contribution. Thus T also has positive flux. The two disjoint sets give two distinct boundary vertices in every sector indexed by J, proving the first bound.

Let $Q = G[N(u)]$, and let H join two root neighbors when their sectors have a crossing edge. Then $Q \subseteq H$, and each endpoint of an edge of $H \setminus Q$ belongs to J. If $G - u$ is connected, projecting a path onto its sectors shows that H is connected. Each of the c components of Q is incident with an edge leaving it, so $|J| \geq c$. □

Theorem 2.2 (Degree-three obstruction)**.** *If a graph has a non-cut interior vertex of degree three, it has at least six boundary vertices. The constant six is sharp.*

Proof. At an interior degree-three root, every nonroot target has at least one closer neighbor. Two closer neighbors would give positive stability, since at most one could be farther. Thus there is exactly one closer neighbor, and at least one farther neighbor: the unique first-step property holds. Its three neighbors are independent. A path of length two among them would give its middle neighbor a positive score at the root. If there is exactly one neighbor edge $v_1 v_2$, then for every $x \in S_1$ the distance from v_2 equals $f(x)$. Interiorness forces the distance from v_3 to equal $f(x) + 1$. No edge can therefore join S_1 to S_3; the same holds for S_2, S_3. This contradicts connectivity of $G - u$. Theorem 2.1 now gives six.

For sharpness, take three internally disjoint paths between two terminals of lengths $1, 4, 4$. The six internal degree-two vertices are boundary; both degree-three terminals have stability zero. Hence $b = 6$. □

The full statement of Theorem 2.2, including extraction of six distinct actual boundary vertices from graph distances and connectivity after deleting the root, is proved in the Lean file `RootSectors.lean`.

3 Interior eccentricity and diameter three

Lemma 3.1 (Eccentricity-two interiors)**.** *If a two-connected graph has at most five boundary vertices and every interior vertex has eccentricity at most two, then its diameter is at most two.*

Proof. Suppose, to the contrary, that the diameter is at least three. Two internally disjoint paths between a diametral pair each have length at least three, so the graph has at least six vertices. Choose a five-element set B containing its boundary and put $I = V \setminus B \neq \varnothing$. Every $u \in I$ has eccentricity two by hypothesis; eccentricity one would make the whole diameter at most two. If two vertices of I were nonadjacent, a closer neighbor of one toward the other would need a farther neighbor at distance three, a contradiction. Thus I is a clique.

Write $H = G[B]$, $S_u = N(u) \cap B$, $t_b = |N(b) \cap I|$, and $F_b = \{c \in B : d(b, c) = 3\}$. For distinct $u, v \in I$,

$$\beta(u, v) = 1 - |S_u \setminus S_v| \leq 0. \tag{2}$$

Hence the S_u are distinct, incomparable, nonempty proper subsets of B. For $b \notin S_u$ the exact score is

$$\beta(u, b) = t_b + |N_H(b) \cap S_u| - |F_b \cap S_u| \leq 0. \tag{3}$$

Also, eccentricity two gives

$$t_b = 0 \quad \Longrightarrow \quad N_H(b) \cap S_u \neq \varnothing \quad \text{for every } u \in I. \tag{4}$$

Assume first $|I| \geq 2$. Every row S_u has at least two elements. If $S_u = \{a\}$, incomparability makes u the unique interior neighbor of a. Any boundary neighbor b of a makes (3) positive at u. Thus a has degree one, impossible.

We next prove that no ordered pair of rows differs by one element. Suppose $S = S_u$, $T = S_v$, and $S \setminus T = \{a\}$. Put $R = S \cap T$, $Y = T \setminus S$, and $Z = B \setminus (S \cup T)$. Both R, Y are nonempty. For every $y \in Y$, all of T is within distance two of y through v, so $F_y \cap S \subseteq \{a\}$. Equation (3) forces

$$t_y = 1, \qquad N_H(y) \cap S = \varnothing, \qquad d(y, a) = 3.$$

Deleting v must not disconnect Y, so some edge joins Y to Z. The partition of five vertices now leaves $|Z| = 1$ or two.

If $Z = \{c\}$, take $y \in Y$ adjacent to c. An interior neighbor w of c would differ from v and would not be adjacent to y. Since $F_y \subseteq \{a\}$, its score toward y is at least $1 + 1 - 1 > 0$. Thus $t_c = 0$. By (4), c has a neighbor in S, which cannot be a because $d(a, y) = 3$; it therefore lies in R. Now c has at least two neighbors in T, one in R and one in Y. But $|T| = 3$, so (3) at v toward c is positive.

If $|Z| = 2$, write $R = \{r\}$, $Y = \{b\}$, $Z = \{c, d\}$. Thus $S = \{a, r\}$, $T = \{b, r\}$. The preceding argument in both directions makes a, b have unique interior neighbors, neither adjacent to r, and $d(a, b) = 3$. Minimum degree two requires each of a, b to have a neighbor in Z; they cannot share one. Relabel so that $a \sim c$, $b \sim d$, and $b \not\sim c$. The score at u toward c forces $t_c = 0$ and $d(c, r) = 3$. Yet (4) at v requires c to be adjacent to b or r, neither of which is possible. We have proved

$$|S_u \setminus S_v| \geq 2 \quad (u \neq v). \tag{5}$$

Every row now has size two or three. A row of size two forces every other row to lie in its three-element complement, which accommodates at most one further row satisfying (5). If all rows have size three, their two-element complements must be pairwise disjoint, again leaving at most two rows. Therefore $|I| \leq 2$.

For $I = \{u, v\}$ there are three possible row types. If S, T are disjoint of sizes two and three, an edge between them makes (3) positive: it supplies one closer boundary neighbor in addition

to the one closer interior neighbor, and leaves at most one farther vertex in the size-two row. No such edge exists, so uv is a bridge. If both rows are disjoint pairs, again there is no edge between them. The fifth boundary vertex c must be adjacent to each pair by (4). Its score at u forces exactly one vertex of S adjacent to c, the other at distance three. The latter has no boundary neighbor and only u as an interior neighbor, giving degree one. Finally, if S, T are triples intersecting at r, let $A = S \setminus \{r\}$, $C = T \setminus \{r\}$. An edge $a \sim c$ between A, C would make the score at u toward c positive; the common vertex r is within distance two of c, so cannot compensate. Hence no such edge exists. Connectivity of $G - u$ and of $G - v$ gives $a \in A, c \in C$ both adjacent to r. The score at v toward a is then at least $1+1+0-1 > 0$. These contradictions exclude two interior vertices.

For $I = \{u\}$, the graph has six vertices. Degree two is excluded by the cycle lemma, degree three by Theorem 2.2, and degree five by the diameter. Thus $\deg(u) = 4$. Let c be its unique nonneighbor. Minimum degree two gives at least two neighbors of c among $N(u)$. To have $\beta(u, c) \leq 0$, exactly two vertices of $N(u)$ must be adjacent to c and exactly two must be at distance three from c. There is no edge between these two pairs. Deleting u disconnects the far pair, the final contradiction. □

Theorem 3.2 (Sharp diameter-three exclusion)**.** *Every two-connected graph of diameter exactly three has at least six Steinerberger boundary vertices. The bound is sharp, as witnessed by C_6.*

Proof. In a diameter-three graph, any vertex of eccentricity three is peripheral and hence boundary: a diametral target has at least one closer neighbor and no farther neighbor. Thus every interior vertex has eccentricity at most two, and Lemma 3.1 gives the contradiction. □

3.1 Independent finite audit and all positive diameter-three graphs

The exclusion above is a hand proof. Independently, the antichain representation gives a finite exhaustive audit: there are 7,579 antichains of nonempty proper subsets of five points, including the empty family, and 1,024 boundary graphs. The maximum antichain size is ten, by counting intersections with the 5! maximal Boolean-lattice chains. Thus every diameter-three five-boundary graph is among exactly 7,760,896 patterns; there is no empirical order cutoff.

Two independently written exhaustive programs reject all two-connected candidates. The C++ program retains every positive labeled graph: 2,466 patterns, giving 71 isomorphism types. Their counts by orders $5, 6, 7, 8, 9, 10$ are $2, 10, 42, 13, 3, 1$. Every nonbridge block has diameter at most two; all reduced graphs occur in the 227-kernel catalogue. The order-ten bound is attained by K_5 with one leaf attached at each vertex. A separate complete Python enumeration adds minimum-degree tests and leaves 518 explicit graphs, of which 238 have the wrong diameter and 280 have a cut vertex. Both programs use integer distances and scores. The positive 71-type classification depends on these finite certificates; the negative diameter-three theorem does not. Neither exhaustive program is claimed to have been verified inside Lean.

4 Boundary metrics and their local profiles

The representation in this section is the frame construction of Müller–Pór–Sereni [4, Lemmas 7–8 and Theorem 13]. We include its short proof to make the completeness of the finite certificates explicit. The new contribution here is the harmonic-boundary exclusion obtained from this representation.

Lemma 4.1 (Boundary profiles)**.** *Let $B = \{b_1, \ldots, b_k\}$ contain the CEJZ boundary of a connected*

graph. Write $M_{ij} = d(b_i, b_j)$ *and* $f(x) = (d(x, b_1), \dots, d(x, b_k))$. *Then*

$$d(x, y) = \|f(x) - f(y)\|_\infty,$$
$$f_i(x) + f_j(x) \geq M_{ij},$$
$$\text{for each } i \text{ there is } j \text{ with } f_i(x) + f_j(x) = M_{ij}.$$

In particular the profiles are injective, and adjacency is forced: $x \sim y$ *if and only if their profile distance is one. Moreover* $\beta(u) = \max_{b \in B} \beta(u, b)$.

Proof. Extend a shortest x–y path by steps that increase the distance from x, until no such step exists. Its final vertex b is CEJZ boundary with witness x, and $d(x, b) = d(x, y) + d(y, b)$. The triangle inequality gives the reverse bound for every coordinate, proving the norm identity. The sum inequalities are triangle inequalities. Extending a shortest b_i–x path in the same way gives the required tight partner b_j. For the last statement extend a shortest u–t path to $b \in B$. For every $w \sim u$,

$$d(w, b) \leq d(w, t) + d(t, b), \qquad d(u, b) = d(u, t) + d(t, b),$$

so $\beta(u, b) \geq \beta(u, t)$. Maximizing proves the identity. □

The difference inequalities $|f_i - f_j| \leq M_{ij}$ also follow from the displayed sum and tightness conditions: choose ℓ tight with i, and use $f_i + f_\ell = M_{i\ell} \leq M_{ij} + M_{j\ell} \leq M_{ij} + f_j + f_\ell$. Thus the two independent implementations may include these inequalities explicitly or derive them implicitly.

Lemma 4.2 (No long pendant metric segment)**.** *In a two-connected graph, let* B *contain its CEJZ boundary and have at least three vertices. For every* $b \in B$,

$$\min_{j \neq k,\ b_j, b_k \neq b} \big(d(b, b_j) + d(b, b_k) - d(b_j, b_k)\big) \leq 1.$$

Equivalently, the usual pendant length of each designated boundary point in its finite metric is zero or one half.

Proof. If that minimum were at least two, let x be any neighbor of b. Its coordinate at b equals one, and $f_j(x) \geq d(b, b_j) - 1$ for all other coordinates. If $f_j(x) \geq d(b, b_j)$, no other coordinate k can be tight with j: for $k \neq b, j$ the assumed inequality gives $f_j(x) + f_k(x) > d(b_j, b_k)$, for $k = b$ this is immediate, and $k = j$ cannot be tight because $f_j(x) > 0$. Thus every $j \neq b$ has $f_j(x) = d(b, b_j) - 1$. Every neighbor of b has the identical profile. Injectivity of profiles forces degree one, contradicting two-connectivity. □

This restriction is independent of the diameter. It bounds the five pendant coefficients used in the uniform finite reduction below.

5 The uniform finite reduction and the core theorem

Theorem 5.1 (Small boundary forces diameter two)**.** *Every finite simple two-connected graph with at most five harmonic boundary vertices has diameter at most two. Equivalently, every two-connected graph of diameter at least three has at least six boundary vertices. The constant six is sharp, for example at* C_6.

The proof below is computer assisted through one finite local certificate. Its parameter reduction applies to every diameter. The only residual metrics have separate hand exclusions, so the large diameter censuses are not dependencies of this theorem.

5.1 The three classical five-point metric cones

For a nonempty proper subset $A \subset \{0,1,2,3,4\}$ let δ_A be the split pseudometric, equal to one when A separates the two arguments and zero otherwise. Let R be the graph metric of $K_{2,3}$ with parts $\{0,1\}$ and $\{2,3,4\}$.

The classical five-point metric fan has 102 maximal cones, in three symmetry classes [5, Section 2]. Up to relabeling, every five-point metric is a nonnegative combination of the five singleton splits and one of these five-element lists:

Type	Nonpendant generators $Q_1, \dots, Q_5$
I	$\delta_{01}, \delta_{12}, \delta_{23}, \delta_{34}, \delta_{40}$
II	$\delta_{02}, \delta_{03}, \delta_{12}, \delta_{13}, R$
III	$\delta_{02}, \delta_{03}, \delta_{14}, \delta_{13}, R$

These explicit expressions are also stated in [6, p. 186]. Write the resulting representation as

$$M = \sum_{i=0}^{4} p_i \delta_{\{i\}} + \sum_{j=1}^{5} q_j Q_j, \qquad p_i, q_j \geq 0. \tag{6}$$

The closed cones include all degeneracies. Indeed generic metrics are dense, only finitely many relabeled cones occur, and every nonnegative coefficient is bounded by the largest distance. A convergent subsequence therefore gives the same expression with zero coefficients permitted.

For an integer metric all ten coefficients are half-integers. The split coefficients are isolation indices, which are halves of finite minima and maxima of integer distance expressions. When present, the coefficient of R is recovered from an entry where $R = 1$ after subtracting the split terms. These facts pass to cone boundaries by continuity. An independent exact arithmetic check in the package verifies that twice the inverse of each of the three 10×10 generator matrices is integral.

The singleton coefficient p_i equals $\frac{1}{2} \min_{j \neq k,\, j,k \neq i}(M_{ij} + M_{ik} - M_{jk})$. One can check this directly in the generator lists: every nonsingleton contribution to each Gromov product is nonnegative, and at every root some pair makes all of them zero. The chosen pairs, in root order $0,1,2,3,4$, may be taken as

I	14	02	13	24	03
II	23	23	01	01	01
III	23	34	01	01	01.

For a boundary metric of a two-connected graph, Lemma 4.2 consequently gives

$$p_i \in \{0, \tfrac{1}{2}\}. \tag{7}$$

5.2 An exact cap preserving all local data

Fix a designated boundary root b, and put

$$D_c = M_{bc}, \qquad h_{ce} = D_c + D_e - M_{ce} \quad (c, e \neq b,\ c \neq e).$$

In its radius-two frame ball, write a profile as

$$f_b = r \in \{0,1,2\}, \qquad f_c = D_c + \epsilon_c, \qquad -r \leq \epsilon_c \leq r.$$

For a positive profile with $r > 0$, Lemma 4.1 becomes

$$\begin{aligned} &D_c + \epsilon_c > 0, \qquad \epsilon_c + \epsilon_e \geq -h_{ce}, \qquad \text{some } \epsilon_c = -r, \\ &\text{for each } c: \quad \epsilon_c = -r \quad \text{or} \quad \epsilon_c + \epsilon_e = -h_{ce} \text{ for some } e \neq b, c. \end{aligned}$$

The complete local data therefore depend only on the ten-entry signature

$$\big(\min(D_c, 3) : c \neq b; \quad \min(h_{ce}, 5) : \{c, e\} \subset B \setminus \{b\}\big). \tag{8}$$

The sums of two offsets lie between minus four and four; values $h > 4$ cannot be tight and impose no further inequality. When $D_c > 2$, positivity is automatic. A designated boundary point c lies in this ball exactly when $D_c \leq 2$. Its shifted coordinates are $r = D_c$, $\epsilon_c = -D_c$, and $\epsilon_e = D_c - h_{ce}$. Triangle inequalities give $h_{ce} \leq 2D_c$, so these values have not been truncated.

Adjacency is the sup-norm-one relation between the shifted vectors. Every signed score toward a designated boundary target is a sum of coordinate differences. Thus the signature determines the profile set, its boundary markings, all adjacencies, and all score coefficients.

For a nonnegative half-integer q, define

$$\operatorname{cap}(q) = \begin{cases} q, & q \leq 2, \\ 5/2, & q > 2 \text{ and } q \notin \mathbb{Z}, \\ 3, & q > 2 \text{ and } q \in \mathbb{Z}. \end{cases}$$

Replace all five q_j in (6) by their caps, leaving the p_i unchanged. Split metrics and R have Gromov products zero or one on distinct triples. Their nonzero distance entries are one or two. Every distance D_c and Gromov product $h_{ce}/2$ is therefore a nonnegative integer combination of the coefficients. Any such quantity at most two is unchanged. If a quantity greater than two changes, one of its contributing coefficients becomes $5/2$ or three, so it stays greater than two. This proves that every signature (8) is preserved.

Every coefficient change is an integer multiplying an integer-valued pseudometric, so the capped metric stays integral. Positive distances stay positive, and diameter greater than two is preserved. There are at most

$$3 \cdot 2^5 \cdot 7^5 = 1,613,472 \tag{9}$$

coefficient representatives to check. Relabelings need not be repeated because the test below is invariant under permutation of the five labels.

No global graph contraction is being assumed. If an actual graph has metric M, each of its boundary radius-two balls transfers into the capped abstract frame ball by $f_c \mapsto f_c + (D'_c - D_c)$, keeping the same offsets. All selected adjacencies, boundary markings, and signed scores are preserved. We do not assert that a graph realizing the entire capped metric exists. Local transfer is all that the proof needs.

5.3 The necessary local model and its certificate

Suppose, toward Theorem 5.1, that $b(G) \leq 5$ and $\operatorname{diam}(G) \geq 3$. Two-connectivity gives at least six vertices. Pad the harmonic boundary to a five-element designated set B if necessary. It still contains CEJZ boundary, and every vertex outside B is interior. By the cycle lemma and Theorem 2.2, these vertices all have degree at least four. All designated vertices have degree at least two.

At each root $b \in B$, select all designated boundary profiles in its radius-two frame ball and any subset of its positive interior profiles. Every actual graph ball must satisfy these conditions:

(i) A selected vertex at distance zero or one from the root has degree at least two if designated, and at least four otherwise.

(ii) Each selected nondesignated vertex at distance one has nonpositive score toward each of the five designated targets.

(iii) If some designated target is at distance at least three from the root, two selected layer-one vertices are joined to two distinct selected layer-two vertices by two edges with disjoint endpoints.

The complete neighborhood of a layer-one vertex is inside this ball, so its degree and scores are exact. For (iii), take two internally disjoint paths from the root to a far target. Stop each at its first layer-two vertex, and shortcut its preceding layer-one vertex directly to the root. This gives the two required edges. Layer-two interior vertices receive no additional constraints; the local test is deliberately only necessary.

The exact finite result is:

Stage	Coefficient representatives
Integral after capping	100,842
With coincident designated points	2,604
Positive metrics of diameter at most two	139
Positive metrics of diameter at least three	98,099
Rejected at at least one boundary root	98,084
Passing all five necessary local tests	15

The surviving coefficients are listed below. Entries are doubled coefficients, in the generator order of (6).

Type	$2(q_1, \ldots, q_5)$	$2(p_0, \ldots, p_4)$	Diameter
I	(1,1,1,1,1)	(1,1,1,1,1)	3
II	(0,0,0,0,3)	(1,1,0,0,0)	4
II	(0,1,1,0,1)	(0,0,1,1,0)	3
II	(1,0,0,1,1)	(0,0,1,1,0)	3
II	(1,1,1,1,0)	(1,1,1,1,1)	3
II	(1,1,1,2,0)	(1,0,1,0,1)	3
II	(1,1,2,1,0)	(1,0,0,1,1)	3
II	(1,2,1,1,0)	(0,1,1,0,1)	3
II	(1,2,2,1,0)	(0,0,0,0,1)	3
II	(2,1,1,1,0)	(0,1,0,1,1)	3
II	(2,1,1,2,0)	(0,0,0,0,1)	3
III	(0,0,0,0,3)	(1,1,0,0,0)	4
III	(0,1,1,0,1)	(0,0,0,1,1)	3
III	(1,0,0,1,1)	(0,0,1,1,0)	3
III	(1,0,1,0,1)	(0,0,1,0,1)	3

This finite step was obtained by exact Boolean/integer constraints and independently reproduced by a C++ program with its own profile construction and exhaustive branching. The 15 complete records agree exactly. The independent run needs 1,141 local types, each with at most fifteen profiles. It supplies 306 explicit feasible selections and 835 explicit contradiction certificates. A separate standard-library Python verifier reconstructs every profile set, checks each selection or contradiction tree, and independently enumerates all coefficient choices using only the verified truth table. No external SAT solver is needed to verify the delivered certificate.

All 835 rejected local systems are also proved as explicit infeasibility theorems in Lean 4.35.0. The 42 proof modules and their aggregate import compile successfully; their axiom audits use only standard logical axioms. An independent semantic comparison checks the generated theorem statements against the separately constructed local profile data. This formalizes the finite contradictions, while cone coverage, capping, and exhaustive parameter coverage retain the mathematical and exact-checker proofs above.

For clarity, the contradiction rules are elementary. Maintain forced and available selected vertices. A vertex with too few available neighbors cannot be selected; a forced vertex with exactly its required number of available neighbors forces all of them. For an interior score, subtract the number of available farther neighbors from the number of forced closer neighbors. If this lower

bound is positive, the vertex is impossible. If it is zero at a forced vertex, every available farther neighbor is forced and every optional closer neighbor is excluded. An impossible two-edge matching is a contradiction; if there are just two available layer-two vertices, both are forced. An undecided vertex may be split into its selected and excluded cases. Each rule preserves every feasible selection, and the certificate checks both children of every split.

5.4 Why the finite check covers unbounded diameters

Every surviving q_j is at most $3/2$. A coefficient larger than two would cap to $5/2$ or three, neither of which appears in a survivor. Hence each surviving representative has a singleton preimage under the cap: the original metric is exactly the surviving capped metric. This rules out the possibility that an arbitrarily large metric was silently identified with a small surviving one.

Thirteen representatives have diameter three and are excluded by Theorem 3.2. The remaining two represent the same metric:

$$M_{ab} = 4, \qquad M_{ac_i} = M_{bc_i} = 2, \qquad M_{c_ic_j} = 3 \quad (1 \le i \ne j \le 3). \tag{10}$$

It has the following short hand exclusion.

Lemma 5.2. *No graph of minimum degree at least two with all of its harmonic boundary in* $\{a, b, c_1, c_2, c_3\}$ *can realize* (10).

Proof. The only possible positive frame profiles are nine vertices $p_{r,i}$, where $r = 1, 2, 3$ and $i = 1, 2, 3$. Their coordinates at a, b are $(r, 4 - r)$; their coordinate at c_i is one and the other two c-coordinates are two. To derive this list, tightness at a, b forces their positive coordinates to sum to four. At most one c-coordinate can be one, since distinct c's have distance three, and tightness forces exactly one to be one and the rest two.

Each row r induces K_3, consecutive rows are completely joined, and the first and third rows have no edges between them. Vertex a sees only the first row, b only the third, and c_i sees just its three column vertices $p_{1,i}, p_{2,i}, p_{3,i}$. If a middle vertex $p_{2,k}$ is selected, choose $j \ne k$. Its score toward c_j equals the number of selected column-j vertices minus one: c_k contributes minus one and each column-j vertex contributes one. Minimum degree two at c_j makes this positive. Thus every middle vertex is absent. Minimum degree two at each c_i then forces all six outer vertices. For $i \ne j$, the profiles of $p_{1,i}$ and $p_{3,j}$ have sup-norm distance two, but these vertices have no common selected neighbor. Their actual distance cannot be two, contradicting Lemma 4.1. □

All possible metrics of diameter at least three have now been excluded, which proves Theorem 5.1. As another independent audit, a fresh full-frame program enumerates all 1,188 subsets arising from the 15 residual representatives (six metric isomorphism types), recomputes actual BFS distances and all scores, and finds no two-connected survivor. This final small check does not use the degree-three obstruction.

6 An exact block formula

A block is a maximal two-connected subgraph, or a bridge regarded as K_2. Let $\mathcal{B}(G)$ be the blocks and $\mathcal{A}(G)$ the cut vertices. Blocks and their incident cut vertices form a tree. In particular, distinct blocks meet in at most one vertex, and every block is isometric in G.

Theorem 6.1 (Block stability formula)**.** *For every* $x \in V(G)$,

$$\beta_G(x) = \max_{B \in \mathcal{B}(G) : x \in B} (\beta_B(x) - \deg_G(x) + \deg_B(x)). \tag{11}$$

All block quantities in this expression are intrinsic to the block.

Proof. Fix a target $y \neq x$. The block-cut tree selects a unique first block B on the route from x to y. If $y \in B$, set $z = y$; otherwise let z be the exit cut vertex of B on that route. For each $w \in B$, every path to y must pass through z, so $d_G(w, y) = d_B(w, z) + d_G(z, y)$. The common tail cancels from the stability sum. Each neighbor of x outside B reaches y through x and contributes -1. Thus

$$\beta_G(x, y) = \beta_B(x, z) - (\deg_G(x) - \deg_B(x)).$$

For $y = x$ the value is $-\deg_G(x)$ and is also obtained by the corresponding within-block expression. Conversely every target $z \in B$ can be used in G, giving the displayed expression. Maximizing proves the formula. □

For a cut vertex x, define

$$t_x = |\{B : x \in B\}|, \quad c_x = |\{B : x \in \partial B\}|, \quad I_x = \mathbf{1}_{\{x \in \partial G\}}, \quad \delta_x = 2(t_x - 1) - c_x + I_x.$$

Theorem 6.2 (Nonnegative block budget)**.** *Every $\delta_x \geq t_x - 2 \geq 0$, and*

$$b(G) = 2 + \sum_{B \in \mathcal{B}(G)} (b(B) - 2) + \sum_{x \in \mathcal{A}(G)} \delta_x. \tag{12}$$

Proof. A vertex belonging to one block has the same stability in that block and in G, by Theorem 6.1. Summing intrinsic block boundaries therefore overcounts each cut vertex by exactly $c_x - I_x$:

$$b(G) = \sum_B b(B) - \sum_{x \in \mathcal{A}(G)} (c_x - I_x).$$

The block-cut tree gives $\sum_x (t_x - 1) = |\mathcal{B}(G)| - 1$. Substitution yields (12). Finally, $c_x \leq t_x$ and $I_x \geq 0$ imply $\delta_x \geq t_x - 2$. □

A nonbridge block has at least three boundary vertices: the known two-boundary theorem says that such a connected graph is a path, which cannot be a nonbridge block. Therefore (12) immediately gives the following.

Corollary 6.3 (The five-boundary reduction)**.** *If $b(G) = 5$, every block has boundary at most five, there are at most three nonbridge blocks, and the possible multisets of their intrinsic boundary sizes are precisely constrained by the following table.*

Multiset	$\varnothing$	(3)	(4)	(5)	(3, 3)	(3, 4)	(3, 3, 3)
Required $\sum_x \delta_x$	*3*	*2*	*1*	*0*	*1*	*0*	*0*

The table is an exact boundary test once the blocks and their incidences are specified. A two-connected block of boundary three is K_3; the four-boundary blocks are already known. The eighteen five-boundary blocks are classified by Theorem 5.1 and Lemma 8.1; the budget is therefore a complete construction rule.

Corollary 6.4 (A block exhausting the boundary)**.** *If G has k boundary vertices and contains a block B with $b(B) = k$, then G consists of B with pendant paths attached at distinct vertices x with $\beta_B(x) = 1$. Conversely every such attachment preserves $b(G) = k$.*

Proof. Equation (12) forces every other block to be a bridge and every defect to vanish. A zero defect implies $t_x = 2$, $c_x = 2$, and $I_x = 0$. The outside components are therefore paths, with no branching and at most one at each attachment point. At such a point, $\beta_G(x) = \max\{\beta_B(x) - 1, 1 - \deg_B(x)\} = \beta_B(x) - 1$; the maximum identity follows also directly from (11). Because the cut vertex is locally boundary and globally interior, its intrinsic stability is exactly one. The same formula proves the converse. □

7 A sharp bound after suppressing bridge paths

Definition 7.1. Let $K(G)$ be the graph obtained by suppressing all degree-two vertices lying on no cycle. These vertices lie internally on paths of bridges. The resulting reduced graph is unique up to isomorphism.

Suppression cannot create a loop or a parallel edge: an alternative connection would put the original path on a cycle. Formula (11) shows that suppression preserves the boundary of every retained vertex; the removed vertices are interior. Thus $b(K(G)) = b(G)$. Conversely arbitrary subdivisions of bridges preserve the boundary cardinality.

Write s for the number of nonbridge blocks and put

$$R = \sum_B (b(B) - 2), \qquad \Delta_* = \sum_x \delta_x, \qquad D = \sum_B (|V(B)| - b(B)).$$

The star in Δ_* distinguishes this defect sum from maximum degree.

Theorem 7.2 (Universal bound retaining block interiors)**.** *For every connected graph with $b(G) = k \geq 2$,*

$$|V(K(G))| \leq 2k - 2 + 2s + D \leq 4k - 6 + D.$$

Proof. Partition the reduced vertices into those in nonbridge blocks, the remaining vertices of degree at least three, and leaves. Call their counts S, A, L. Then $S \leq R + 2s + D$. A vertex counted by A belongs only to bridge blocks, has $t_x = \deg(x) \geq 3$, and has $c_x = t_x$, $I_x = 0$; hence $\delta_x = t_x - 2 \geq 1$. Thus $A \leq \Delta_*$, while $L \leq k$. Using $R + \Delta_* = k - 2$ gives $S + A + L \leq 2k - 2 + 2s + D$. Finally $s \leq R \leq k - 2$. □

Theorem 7.3 (Sharp finite kernel)**.** *Suppose every nonbridge block B satisfies at least one of:*

(i) $\partial B = V(B)$*;*
(ii) $\operatorname{diam}(B) \leq 2$*.*

If $b(G) = k \geq 2$, then

$$|V(K(G))| \leq 4k - 6. \tag{13}$$

Equality holds precisely when $K(G)$ is obtained from a tree with k leaves and all internal vertices of degree three by replacing every internal vertex by a triangle, attaching its three incident edges at three distinct triangle vertices. At $k = 2$, this means a single edge.

Proof. For every nonbridge block in the stated class,

$$|V(B)| \leq 3(b(B) - 2), \tag{14}$$

with equality only for K_3. In case (i), $|V(B)| = b(B) \geq 3$. In case (ii), the diameter-two boundary theorem gives $|V(B)| \leq b(B) + 1$. The three-boundary two-connected case is K_3; for $b(B) \geq 4$, the inequality $b(B) + 1 < 3(b(B) - 2)$ is strict.

Use the partition S, A, L from Theorem 7.2. Then

$$|V(K(G))| = S + A + L \leq 3R + \Delta_* + k = 4k - 6 - 2\Delta_* \leq 4k - 6.$$

Equality forces $\Delta_* = 0$, equality in every block inequality, no overlap between nonbridge blocks, $A = 0$, and $L = k$. Consequently all nonbridge blocks are pairwise disjoint triangles; all noncyclic vertices are leaves. Every triangle vertex must be a cut vertex, since otherwise it is a non-leaf boundary vertex. Zero defect forces exactly one bridge at each triangle vertex. Contracting the triangles produces the stated trivalent tree.

Conversely, a trivalent tree with k leaves has $k - 2$ internal vertices. Its triangle expansion has $3(k - 2) + k = 4k - 6$ vertices. Formula (11) gives stability zero at every triangle vertex and one at every leaf. Its boundary therefore has exactly k vertices. □

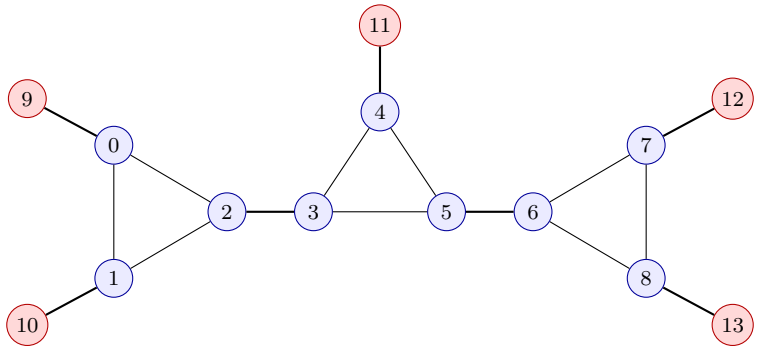


Figure 1: The unique 14-vertex reduced equality graph at boundary five. Red vertices are exactly the boundary. The labels agree with the Lean certificate `Sharp14.lean`. Any bridge may be subdivided arbitrarily.

Proposition 7.4 (A degree constraint in the same class)**.** *Under the hypotheses of Theorem 7.3,*

$$\deg_G(x) + \mathbf{1}_{\{x \in \partial G\}} \le b(G) \qquad (x \in V(G)).$$

By Theorem 5.1, every five-boundary graph belongs to this class. Thus every such graph has maximum degree at most five, and every boundary vertex has degree at most four.

Proof. Each block satisfies $\deg_B(x) + \mathbf{1}_{\{x \in \partial B\}} \le b(B)$. This is immediate in case (i). In case (ii), either every vertex is boundary, or there is one universal interior vertex. That vertex has degree $b(B)$, and every other vertex is nonuniversal and has degree at most $b(B) - 1$. For a non-cut vertex, use $b(B) \le b(G)$. At a cut vertex retain just the incident blocks and its own defect in the nonnegative budget to obtain

$$b(G) \ge 2 + \sum_{B \ni x} (b(B) - 2) + \delta_x = \sum_{B \ni x} b(B) - c_x + I_x \ge \deg_G(x) + I_x.$$

□

8 The complete 227-kernel classification

By Theorems 6.2 and 5.1, every nonbridge block of any five-boundary graph has diameter at most two. There is now no restriction to a subclass. For consistency with the supplied initial catalogue files, let $\mathcal{C}_2$ denote the class whose nonbridge blocks have diameter at most two; all graphs considered here belong to it.

Lemma 8.1 (The finite block palette)**.** *Exactly 24 nonbridge block types can occur in a connected graph with boundary at most five. Their orders and intrinsic boundaries are:*

Order	*Boundary*	*Types*	*Description*
3	*3*	*1*	K_3
4	*4*	*3*	C_4, K_4 *minus an edge,* K_4
5	*4*	*2*	*cones over* P_4 *and* C_4
5	*5*	*8*	*all-boundary two-connected graphs of order five*
6	*5*	*10*	*cones over connected order-five graphs with no universal vertex*

The eight order-five cores are

$$C_5, \quad \Theta(1,2,3), \quad K_{2,3}, \quad K_2 \vee \overline{K_3}, \quad K_{2,3} + e_3, \quad K_5 - E(P_3), \quad K_5 - e, \quad K_5.$$

Here e_3 joins two vertices in the three-vertex part of $K_{2,3}$, and $E(P_3)$ denotes two adjacent edges. The ten order-six cores are $K_1 \vee H$, where H runs through the connected five-vertex graphs of maximum degree at most three. These descriptions contain no boundary predicate that still needs to be classified. The adjacency lists and all stability values are in `small_biconnected_cores.json`.

Proof. The budget bounds the block boundary by five; the diameter-two theorem then bounds its order by six. When there is an interior vertex, it is unique and universal, so the graph is a cone. Two-connectivity is equivalent to connectivity after deleting that vertex, and uniqueness says that the base has no universal vertex. These facts give the listed descriptions. The remaining finite counts were certified by enumerating all 33,864 labeled graphs of orders three through six, with exhaustive degree-compatible permutation tests for isomorphism. This checker is independent of the atlas and of nauty. □

Remark 8.2. Each boundary vertex in a two-connected cone with a unique universal vertex has stability at least two. Indeed its base is connected, and every base vertex u has a nonneighbor at base distance two. Such a target has both the universal vertex and a base neighbor as closer neighbors of u; all other neighbors are at distance at most two. Hence the ten order-six/five-boundary cores admit no boundary-preserving pendant attachment by Corollary 6.4.

Theorem 8.3 (Exact finite classification)**.** *A finite simple connected graph G has five boundary vertices if and only if it is obtained by independently subdividing bridges of one of the 227 pairwise nonisomorphic graphs in* `reduced_kernels_boundary5.g6`. *Their order distribution is*

Order	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*Total*
Kernels	*10*	*33*	*34*	*42*	*41*	*34*	*19*	*10*	*3*	*1*	*227*

The unique order-fourteen graph is Figure 1.

Mathematical completeness and finite verification. Theorem 7.3 bounds each reduced graph by 14 vertices, and Lemma 8.1 gives its complete nonbridge palette. Include K_2 as the bridge block. Starting from these 25 seeds, perform the following finite closure: attach a rooted seed at a vertex of a current graph, or at a newly inserted point inside one of its bridges; suppress any new degree-two bridge vertex; retain boundary at most five; and identify isomorphic graphs.

This procedure is exhaustive, with single-block graphs as the induction base. Remove an end block from a target reduced graph, retaining its attachment vertex x. If a graph remains, it is connected and stays in $\mathcal{C}_2$. Identifying graphs A, B at a vertex gives

$$b(A \cup B) = b(A) + b(B) - I_A - I_B + I_{A \cup B} \geq b(A),$$

because $b(B) \geq 2$. Thus end-block deletion cannot increase the boundary. Only x can newly require suppression. Reversing deletion attaches the block at an existing vertex or at a point in a bridge, exactly the two operations above. Induction on the number of blocks proves completeness.

The computed closure contains 264 kernels with boundary at most five: 1, 5, 31, and 227 at boundaries 2, 3, 4, and 5. A second run without orbit pruning reproduced the same set. A separate verifier used fresh breadth-first distances and exact graph isomorphism, without nauty or the incremental stability formula. It verified all seeds, pairwise nonisomorphism, and closure under 7,788 attachment placements; all 3,194 placements retaining boundary at most five were already in the catalogue. The completeness argument plus these finite checks proves the count. Finally, bridge subdivision preserves the boundary, completing both directions. □

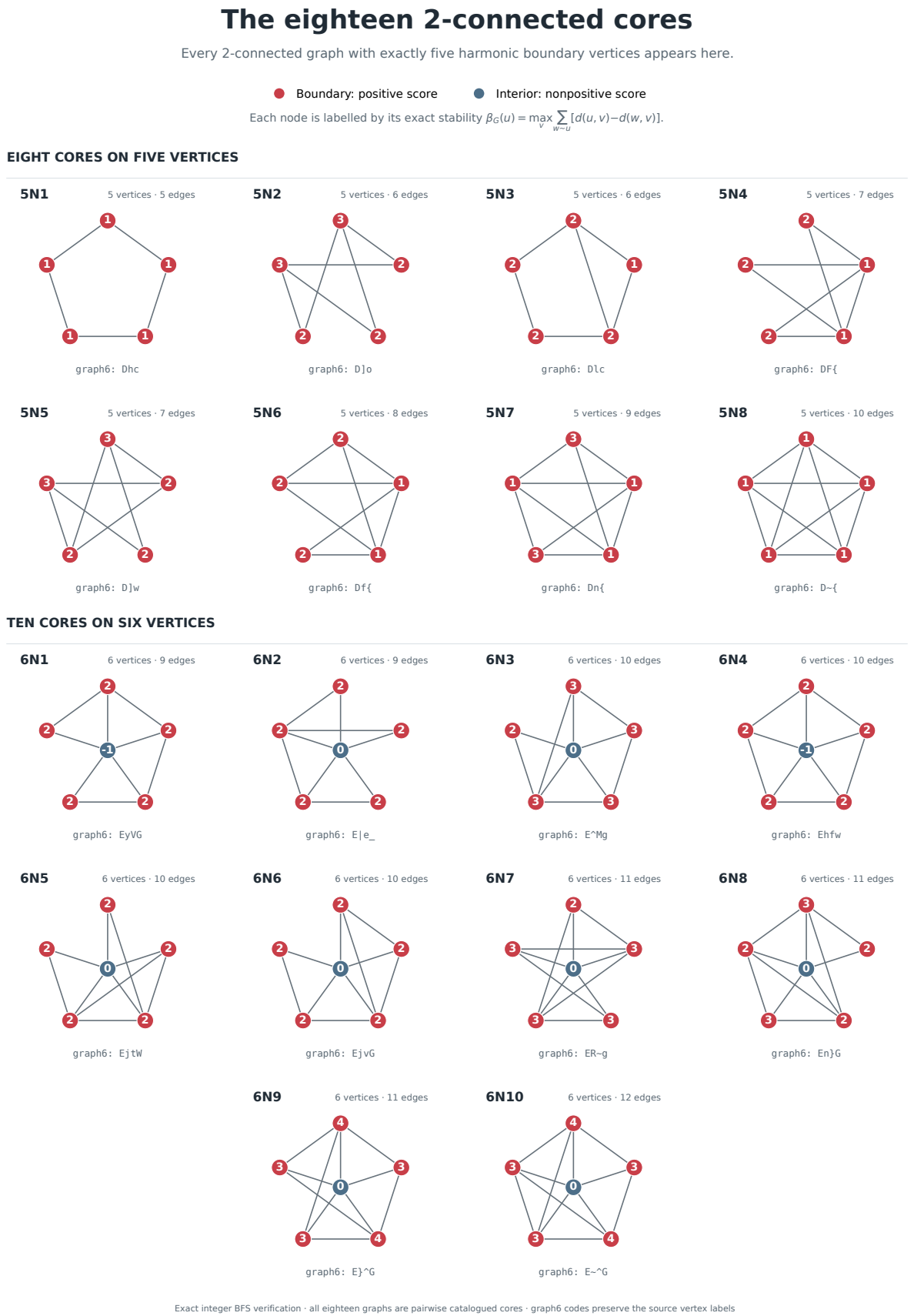


Figure 2: All eighteen two-connected cores at boundary five. Vertex labels are exact stability values, not vertex identifiers. Graph6 codes and the full adjacency data are provided with the figure.

9 All five-boundary cacti

For a cactus, every nonbridge block is a cycle. The intrinsic stability of every vertex is one on an odd cycle and two on an even cycle. To see this, compare a vertex with a target around the cycle: an even antipode has two closer neighbors; an odd farthest target has one closer and one equidistant neighbor. No target produces a larger value.

Theorem 9.1 (Cactus boundary and exact classification)**.** *The boundary of a cactus consists exactly of leaves, degree-two vertices on cycles, and vertices incident with one even-cycle block and one bridge. If*

$$R_C = \sum_{\text{cycle blocks } C} (|C| - 2), \qquad X = \sum_{x \in \mathcal{A}(G)} (t_x - 2),$$

and E is the number of vertices meeting exactly one even cycle and one bridge, then $b(G) = 2 + R_C + X + E$. Consequently the following is a complete necessary-and-sufficient classification at boundary five:

Cycle multiset	*Required attachment condition*
No cycles	$X = 3$ *(trees with five leaves)*
One triangle	$X = 2$
Two triangles	$X = 1$
Three triangles	$X = 0$
One square	$X + E = 1$
One pentagon	$X = 0$
One triangle and one square	$X = 0$, $E = 0$

Every condition is computed on the actual cactus block tree. All bridge-path lengths are arbitrary. Equivalently, there are exactly 121 reduced cactus kernels, supplied in `cactus_reduced_kernels_boundary5.g6`.

Proof. A bridge block has intrinsic stability one at each endpoint. Combining these values and the cycle values with (11) gives the boundary list. Every block vertex is intrinsically boundary, so $c_x = t_x$. The only globally boundary cuts are the E exceptional cuts, and $\delta_x = t_x - 2 + I_x$. Equation (12) gives the count.

The condition $R_C + X + E = 3$ is necessary and sufficient. Since every cycle has length at least three, partitioning $R_C \le 3$ gives exactly the seven rows. It also forces every cycle to have length at most five, so every five-boundary cactus lies in $\mathcal{C}_2$. Selecting the cactus members of the complete catalogue in Theorem 8.3 gives 121 kernels. □

For clarity, the square row permits exactly one pendant path, or two pendant paths at the same square vertex. The pentagon permits a single pendant path at any chosen subset of its vertices. In the triangle-and-square row, the cycles share one vertex directly; pendant paths may occur only at the other two triangle vertices. The triangle-only rows are described without case ambiguity by X. The unique 14-vertex kernel is a cactus, so the bound is sharp already in this subclass.

10 A sharp theorem for every two-tree

A k-tree starts with K_{k+1} and repeatedly adds a vertex whose neighbors are the vertices of an existing k-clique. In particular, a two-tree is built by adding triangle ears on edges. This section has no diameter restriction.

Theorem 10.1 (Exact minimum boundary of a two-tree)**.** *Among all two-trees on n vertices, the minimum boundary cardinality is*

$$\min b(G) = \begin{cases} 3, & n = 3, \\ 4, & n = 4, 5, \\ \lceil n/2 \rceil + 2, & n \geq 6. \end{cases}$$

In particular, $n \leq 2b(G) - 4$ for every two-tree of order at least six. There are exactly three two-trees with five boundary vertices: the book of three triangles $K_2 \vee \overline{K_3}$, the cone over P_5, and the cone over the five-vertex tree of degree sequence $(3, 2, 1, 1, 1)$.

The order bound is sharp at every even order, and its rounded boundary form is sharp at every odd order as well. Maximality matters: there are two-connected series-parallel graphs on nine vertices with boundary six, so the same order bound does not hold for all partial two-trees.

10.1 Localizing stability in chordal graphs

A graph is chordal if it has no induced cycle of length at least four. We use the elementary chordal facts that minimal vertex separators are cliques and that a perfect elimination ordering can end in any specified clique.

Theorem 10.2 (Radius-two localization)**.** *In every connected chordal graph,*

$$\beta_G(u) = \max_{y : d_G(u,y) \leq 2} \beta_G(u, y).$$

This identity concerns the maximum stability itself, not just its sign.

Proof. Let $y \notin N[u]$, let C be its component in $G - N[u]$, and put $S = N(C) \subseteq N(u)$. For every $s \in S$ a path u, s, C, y avoids $S \setminus \{s\}$, so S is a minimal u–y separator and is a clique. There is a vertex $x \in C$ adjacent to all of S. For completeness, take a perfect elimination ordering of $G[S \cup C]$ ending in S. Deleting a simplicial vertex of C preserves connectivity of the remaining C, and preserves its full neighborhood S: a neighbor $s \in S$ of the deleted vertex is also adjacent to its remaining neighbor in C. The last vertex of C is therefore complete to S.

Set $r = \min_{s \in S} d(s, y)$ and $A = \{s \in S : d(s, y) = r\}$. Since S is a clique, its other members have distance $r + 1$. Every path from $N(u)$ to y enters C through S. It follows that the distances of vertices of $N(u)$ to y are r on A, $r + 1$ on $(N(u) \cap N[A]) \setminus A$, and $r + 2$ on $N(u) \setminus N[A]$. As $d(u, y) = r + 1$,

$$\beta(u, y) = |A| - |N(u) \setminus N[A]| \leq |S| - |N(u) \setminus N[S]| = \beta(u, x).$$

The final equality uses $N(x) \cap N(u) = S$, and $d(u, x) = 2$. □

Lemma 10.3 (Interior degree in a chordal graph)**.** *In a k-connected chordal graph, $k \geq 2$, every nonuniversal interior vertex has degree at least $2k + 1$.*

Proof. First $G[N(u)]$ is connected. A path in $G - u$ between two neighbors of u can be shortened into $N(u)$, because each excursion through a component of $G - N[u]$ enters and exits its clique separator. Choose such a component, with separator S of size $s \geq k$, and the vertex x complete to S from the preceding proof. Put $T = N(u) \cap N[S]$. Interiorness gives $0 \geq \beta(u, x) = s - \deg(u) + |T|$, so $\deg(u) \geq s + |T|$. The set S is proper in $N(u)$, since otherwise that score is positive. Connectivity of $G[N(u)]$ gives $|T| \geq s + 1$, proving the result. □

For an edge ab of a two-tree, put $t_{ab} = |N(a) \cap N(b)|$. The neighbor graph $G[N(u)]$ is a tree; the degree of its vertex v is t_{uv}. This follows directly by induction under triangle-ear addition. Each exterior component has a separator which is an edge ab of this tree; such an attachment exists exactly when $t_{ab} \geq 2$. Consequently Theorem 10.2 gives the exact formula

$$\beta(u) = \max \begin{cases} 2 + t_{uv} - \deg(u) & (v \in N(u)), \\ 2 + t_{ua} + t_{ub} - \deg(u) & (uab \text{ a triangle}, \ t_{ab} \geq 2). \end{cases} \tag{15}$$

Indeed, the closed neighborhood of the edge ab in the neighbor tree has size $t_{ua} + t_{ub}$. A common exterior neighbor of a, b gives the second score; a neighboring target v gives the first.

10.2 Pruning and the sharp count

Lemma 10.4 (Pruning)**.** *Let G be a noncomplete k-tree, $k \geq 2$, and remove any set Q of degree-k vertices. If $H = G - Q$ has at least $k + 2$ vertices, every simplicial vertex of H is a boundary vertex of G.*

Proof. Degree-k vertices of a noncomplete k-tree are independent and simplicial. Their removal leaves a k-tree while at least $k+1$ vertices remain. For a simplicial u of H, the k-clique $S = N_H(u)$ extends in $H - u$ to a clique $S \cup \{x\}$. Thus $x \notin N_H[u]$ and $d_G(u, x) = 2$. Every removed neighbor q of u has its other $k - 1$ neighbors in S. Since $k \geq 2$, $d_G(q, x) = 2$. The k vertices of S are closer and all other neighbors are equidistant, so $\beta_G(u, x) = k > 0$. □

If the remainder is K_k, all its vertices are universal. If it is K_{k+1} and all its vertices were interior, a removed vertex q omits one core vertex u and gives the same positive score $\beta_G(u, q) = k$. These observations handle the small remainders below.

Proof of the lower bound in Theorem 10.1. Suppose $n \geq 6$, let $I = V \setminus \partial G$, let $i = |I|$, and write $I_5 = \{u \in I : \deg(u) = 5\}$. Every interior vertex has degree at least five, by Lemma 10.3, or by $\deg(u) = n - 1 \geq 5$ when it is universal.

We assign a distinct boundary vertex to every $u \in I_5$. Its neighbor tree has five vertices. It cannot be a star, because the star center gives a positive first score in (15). If the neighbor tree is P_5, the second score says that only its two terminal edges can have exterior attachments: an active edge's endpoint degrees must sum to at most three. Its middle vertex c therefore has global degree three. If the neighbor tree is the fork of degrees $(3, 2, 1, 1, 1)$, only the edge from the degree-two vertex to its leaf can have an exterior attachment. The branching vertex c has global degree four. In both cases c is boundary by Lemma 10.3; assign it to u.

This assignment is injective. The two types cannot collide because their assigned vertices have different global degrees. In the path case, write the neighbor path as a, b, c, d, e. The only alternative interior neighbors that could assign c are b, d. In either of their neighbor trees, c has degree one, rather than the degree two required of a path middle. In the fork case, two other neighbors of the assigned branch have global degree two. The sole remaining possible interior neighbor again sees the assigned vertex with link degree one, rather than three.

Set $E_B = \sum_{v \in \partial G} (\deg(v) - 2)$. Injectivity gives $E_B \geq |I_5|$: path assignments use one unit, and fork assignments two. The degree sum $4n - 6$ now yields

$$2(b - i - 3) = E_B - |I_5| + \sum_{u \in I : \deg(u) \geq 7} (\deg(u) - 6) \geq 0. \tag{16}$$

Equality would force all assignments to be path assignments, and every boundary vertex of degree greater than two to be an assigned degree-three middle. Each such middle has three

neighbors of degree greater than two: its assigned interior root and its two neighboring vertices in the path.

Delete all degree-two vertices. They are independent, and the remainder H has at least two vertices, because each deleted vertex has two undeleted neighbors. Every remaining boundary vertex still has degree three in H. If H is a noncomplete two-tree, its simplicial degree-two vertex cannot be boundary, but cannot be interior by Lemma 10.4. If $H = K_3$, all its vertices would be interior, excluded by the small-remainder argument. If $H = K_2$, both vertices are universal and boundary, again impossible. Equality in (16) is therefore excluded. Its integer right side is even, hence at least two. Thus $b - i \geq 4$, and $n = b + i \leq 2b - 4$. □

10.3 Matching constructions and classification at five

For $r \geq 2$, start with a path $v_1, \ldots, v_r$ and independent vertices $c_1, \ldots, c_r$. Join c_j to v_j and its available path neighbors v_{j-1}, v_{j+1}. This core is a two-tree. At each endpoint v, with its corresponding c, choose either of the following three-vertex-end configurations (the vertex c is already present):

- add p on vc, and then q on vp;
- add p, q separately on vc.

Every v_j has degree five; its neighbor tree is a path or the permitted fork. Formula (15) gives $\beta(v_j) = 0$. Every other vertex has degree at most four and is boundary. Thus $n = 2r + 4$ and $b = r + 4$. The three unordered choices of endpoint types give three equality examples for each $r \geq 2$. The following sections prove that these exhaust the even-order equality cases, classify all odd-order cases, and give their exact unlabelled counts. For $r = 1$ the two appropriate choices give the cones over P_5 and the fork, of order six. To obtain every odd order, use the first type at both ends and add one ear on the outer edge pq. This activates only a permitted terminal edge in the adjacent interior vertex's link. The old p, q have stability at least two, so remain boundary, and exactly one new boundary vertex is added. This proves sharpness at every order.

At boundary five the bound forces $n \leq 6$. The two two-trees of order five are the three-triangle book and the cone over P_4, of boundaries five and four respectively. At order six there is one interior vertex; its degree is five, so it is universal. A two-tree with a universal vertex is the cone over a tree. The only five-vertex trees without a universal vertex are P_5 and the fork. This proves the stated classification.

Proposition 10.5 (All k-trees at boundary five)**.** *Every k-tree, $k \geq 2$, satisfies*

$$n \leq (k + 1)(b - k),$$

with equality only for K_{k+1}. Across all $k \geq 2$, the five-boundary graphs are the three two-trees above, K_5 minus one edge (a three-tree), and K_5 (a four-tree). For $k = 1$ they are all trees with five leaves.

Proof. When $n \geq 2k + 2$, interior degrees are at least $2k + 1$ and boundary degrees at least k. The degree sum $2kn - k(k + 1)$ gives the bound. There are at least two boundary vertices of degree greater than k: delete all degree-k vertices and apply Lemma 10.4 to the two simplicial vertices of a noncomplete remainder. For a complete remainder, at most one boundary core vertex would force every deleted vertex to omit that same vertex, leaving at least two universal interior vertices, a contradiction. Hence the large-order bound improves by two.

For $n \leq 2k + 1$, minimum degree k forces diameter at most two. The known diameter-two theorem gives $b \geq n - 1$. The cases $n = k + 1, k + 2$ are a clique and a clique minus one edge; in the latter there are at least two universal vertices and $b = n$. For $n \geq k + 3$, $(k + 1)(n - k - 1) - n = kn - (k + 1)^2 \geq k - 1 > 0$. This proves the bound and its equality statement. For $b = 5$, it leaves

$k = 2, 3, 4$ and orders at most $9, 8, 5$. Independent ear generation and recursive simplicial-degree-k recognition on those finite ranges give exactly the five listed graphs. The two-tree part has already been proved directly above. □

11 Equality in the two-tree bound and quartic growth

Write $b = |\partial G|$, $i = |V(G) \setminus \partial G|$, and $e = b - i - 4 = 2b - |V(G)| - 4$. This section strengthens the sharp two-tree bound $b \geq \lceil n/2 \rceil + 2$ for $n \geq 6$. Its most pronounced consequence is a difference between the two parities.

Theorem 11.1 (Complete equality grammars and growth)**.** *Let a_n be the number of isomorphism classes of two-trees on n vertices attaining $b = \lceil n/2 \rceil + 2$. Then $a_6 = 2$ and $a_n = 3$ for every even $n \geq 8$. At odd orders,*

$$a_n \sim n^4/192.$$

More explicitly, for every odd $n \geq 19$,

$$a_n \geq \frac{1}{8}\binom{(n-19)/2+4}{4}.$$

All odd extremizers with noncomplete degree-two-pruned core are given by an explicit finite grammar with 27 states, 83 productions, and 194 terminal choices. Exactly two additional complete-core examples occur, both at order seven. Every extremizer, of either parity, has maximum degree at most six.

The exact rational generating function and period-eight formula are proved in the subsequent exact-enumeration section. The argument below retains a separate proof of the quartic growth order.

The finite grammar is a necessary-and-sufficient description at every order, rather than a list inferred from an initial census. Its graph-theoretic semantics are proved below; independent integer checkers regenerate the complete state closure and certify the pruning. The grammar data are in `checks/extensions/two_tree_equality/`.

11.1 A structural defect identity

Use the injective assignment from degree-five interior vertices to boundary vertices established in the sharp-bound proof. Path links are assigned their degree-three middle and fork links their degree-four branch. Let f count fork assignments and let U be the unassigned boundary vertices. The exact degree identity is

$$2e + 2 = f + \sum_{x \in U} (\deg x - 2) + \sum_{\substack{x \notin \partial G \\ \deg x \geq 7}} (\deg x - 6). \tag{17}$$

All terms are nonnegative.

Delete all degree-two vertices and call the resulting graph H. If H is noncomplete, it is a two-tree. Every degree-two vertex of H is boundary in G, by the pruning lemma. It cannot be an assigned path middle, because all three neighbors of such a middle have original degree greater than two. Each core ear therefore consumes at least one distinct unit of (17). Since H has at least two ears, this gives

$$\begin{aligned} &\#\{x : \deg_H x = 2\} \leq 2e + 2, \\ &\deg_G x \leq 2e + 6 \qquad (x \notin \partial G), \\ &\deg_G x \leq \max(4, 2e + 3) \quad (x \in \partial G). \end{aligned} \tag{18}$$

For the last bound, an unassigned boundary core ear leaves one unit for another core ear; an unassigned boundary non-ear leaves two. Assigned boundary vertices have degree at most four.

The triangle–edge incidence graph of a two-tree is a bipartite tree. Remove its edge-nodes of degree one. For noncomplete H, the leaves of the remaining tree correspond exactly to its ears. Thus its total branching excess is at most $2e$.

There are elementary complete-core exceptions. If $H = K_2$, all vertices are boundary and $n = e + 4$. If $H = K_3$, write p, q, r for the numbers of attached ears on its three edges. At least two numbers are positive. A core vertex is interior exactly when its opposite multiplicity is zero and both incident multiplicities are positive. Consequently, three positive multiplicities give $n = e + 4$ and $p + q + r = e + 1$, all boundary; exactly two positive multiplicities give $n = e + 6$ and $p + q = e + 3$, with one interior vertex. Retain only cases with $n \geq 6$. At $e = 0$ this gives $(p, q, r) = (1, 2, 0)$, and at $e = 1$ it gives $(1, 3, 0)$ and $(2, 2, 0)$, up to permutation.

11.2 Even equality: precisely three strips

At $e = 0$, both units of (17) are consumed by two core ears, so the internal incidence tree of H is a path. All interior degrees are five or six and all boundary degrees are at most four.

For $r \geq 2$, start with a path $v_1, \ldots, v_r$ and independent vertices $c_1, \ldots, c_r$. Join c_j to v_j and every existing path neighbor of v_j. At either endpoint v, with corresponding c, choose a cap:

- P: add p on edge vc, then q on edge vp;
- F: add p and q separately on edge vc.

These give $PP(r), PF(r), FF(r)$, with $2r + 4$ vertices. The r path vertices are interior of degree five; all others are boundary. Their numbers of degree-four boundary vertices are respectively zero, one, and two, so they are pairwise nonisomorphic.

For completeness, process the triangle path H from one end. For each active endpoint record current degree and accumulated local boundary requirement, and record the final triangle multiplicity of the active edge. The exact local formula is

$$\beta(u) = R_u - \deg u, \qquad R_u = \max\{2 + t_{uv},\ 2 + t_{ua} + t_{ub} : uab \text{ a triangle},\ t_{ab} \geq 2\}. \tag{19}$$

Here t_{uv} is the number of triangles on edge uv. Requirements can be capped at seven. Forget a boundary vertex of degree three or four with weight $+1$, or an interior vertex of degree five or six with weight -1; each added ear has weight $+1$. Accepted weight is exactly $b - i$.

The complete closure has 18 states and 58 ordinary parameterized transitions. Exact prefix and suffix potentials show that weight four paths use only these four states and arcs:

	$((d_u, R_u), (d_v, R_v), t)$	initial	terminal
A	$((2,5),(3,4),2)$	2	2
B	$((2,5),(4,4),2)$	–	1
C	$((2,5),(3,5),2)$	–	2
D	$((2,5),(4,5),2)$	3	1

$$A \longrightarrow B\ (+1), \quad B \longrightarrow C\ (-1), \quad C \longrightarrow D\ (+1), \quad D \longrightarrow C\ (-1).$$

The initial and final choices give the P, F caps. More explicitly, in state C the active edge is (c_j, v_j) of degrees $(2, 3)$. The step to D adds v_{j+1}, forgets c_j as a degree-three boundary vertex, and retains (v_{j+1}, v_j) of degrees $(2, 4)$. The return step adds c_{j+1}, forgets v_j as a degree-five interior vertex, and retains (c_{j+1}, v_{j+1}). Thus the cycle is exactly strip extension. The two order-six exceptions are the cones over the five-vertex path and the five-vertex fork. This proves the full even classification.

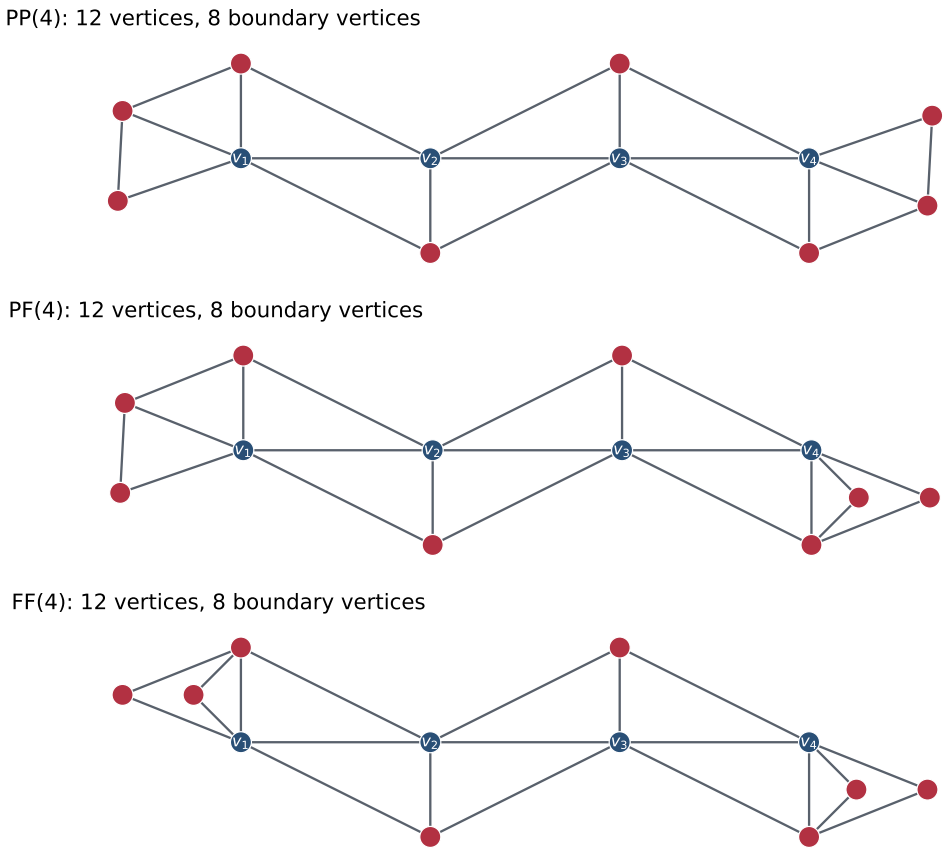


Figure 3: The three even extremal families at $r = 4$. Blue vertices are the four degree-five interior vertices; red vertices form the eight-point boundary.

11.3 An exact rooted grammar at odd equality

For $e = 1$, (18) initially bounds interior degree by eight, boundary degree by five, and core leaves by four. A rooted component has a pending root edge uv and at least one outside core triangle. Its root ears are deferred. If k is the number of inside core triangles on that edge, record at each endpoint a triple (d, A, B) whose requirement, at final root multiplicity $t \geq 2$, is

$$\rho(t) = \max(A, B + t).$$

The state is $(k; d_u, A_u, B_u; d_v, A_v, B_v)$, with endpoint triples sorted. Continuing degrees are at most seven; cap A at nine and B at seven. These caps preserve all decisions at final degree at most eight. A rooted component has one, two, or three leaves.

This information is sufficient: a forgotten vertex has every incident edge closed. Its only possible dependence on the pending root edge is as the opposite edge of a triangle, whose active condition is merely $t \geq 2$ and is already fixed. The two retained requirements remain maxima of a constant and a constant plus t.

The operations are: an initial leaf triangle with ears on its closed edges; adding one triangle and retaining a new root edge; joining two components at a triangle; and identifying two pending root edges. In the last operation degrees add with subtraction of one, because the root neighbor is counted twice, while k, A, B combine by addition, maximum, maximum, respectively. In a triangle join, the degree of the identified vertex is the sum of its child degrees without subtraction. All formulas follow directly from (19) and are given explicitly in the supplied note. Final gluing identifies the two pending roots, adds any root ears, and finalizes both endpoints. Every forgotten vertex is required to be boundary of degree three through five or interior of degree five through eight. Its sign, together with the ears, contributes to component weight; the two pending roots are excluded.

Rooting the internal incidence tree at an internal edge yields two components, each with at most three leaves. Recursion into that tree gives exactly the four operations. This proves necessity; the same interpretation and local formula prove sufficiency. When the numerical root triples coincide, either physical ordering is permitted: equal data do not imply a swapping automorphism. No final productive equality state has such a tie.

The complete raw closure has $88, 985, 1153$ states with one, two, three leaves, respectively. It has nine initial, 126968 binary, and 6030 unary productions. Let $\mu(s)$ be the minimum component weight. The certified potentials satisfy, for every production,

$$\sum_j \mu(s_j) + w - \mu(s) \geq 0.$$

Rank-decreasing productivity witnesses attain every minimum. A minimum completed gluing is an actual graph and has weight at least four. Thus any accepted weight-five derivation has slack at most one above its component minima. Adding a slack coordinate 0 or 1, retaining productive states, and tracing backwards from weight-five terminal choices gives precisely 27 states, 83 productions, and 194 terminal choices. The two complete-core exceptions at order seven finish the classification.

An independent verifier regenerates all raw rules and all 113830 valid terminal gluings, checks 133007 potential inequalities, and checks the rank certificates and exact backward pruning. A separate program realizes all 4386 productivity witnesses as actual graphs and recomputes their full distance boundary scores by BFS. Both checks pass. Every interior degree appearing in an accepted rule or terminal choice is five or six. This proves the claimed uniform degree-six bound.

11.4 Five strip parameters suffice

The compact odd grammar has no triangle-join production. Its pending root multiplicities satisfy $k \le 2$, and final root core multiplicity is at most three. Thus the internal core incidence tree branches only at trivalent edge-nodes, with at most two branches.

Number the compact states from zero in the order in the data file. The only cyclic strongly connected components of its unary part are

$$\{1,4\}, \quad \{2,5\}, \quad \{10,11\}, \quad \{22,23\}, \quad \{6\}, \quad \{12\}.$$

Each state has exactly one rule staying inside its cyclic component. The two-step cycles insert a degree-five strip pair; the self-loops insert a degree-six core vertex together with an ear. Every complete cycle has weight zero and adds exactly two vertices.

After contracting these components, the dependence graph is acyclic; binary rules increase leaf count. Consequently derivations consist of finitely many templates, with one repetition parameter for each cyclic component occurrence. To count occurrences rather than just topological core paths, assign to each contracted component C

$$M(C) = \mathbf{1}_{C \text{ cyclic}} + \max\Big\{\sum_j M(C_j) : \text{a production enters } C \text{ from children outside } C\Big\}.$$

The exact per-state vector is

$$(0,1,3,0,1,3,2,0,1,2,4,4,3,3,0,2,\\ 0,1,2,1,2,3,5,5,4,3,4).$$

If the core branches, root at a trivalent edge-node, so the terminal choice has $k_1 + k_2 = 3$. The maximum sum of these two state values is five. If the core is a path, root next to an end triangle: that side is an initial leaf, with no repetition parameter, and the other side has at most three. Thus every odd extremizer has a finite-template description with at most five nonnegative repetition parameters. Each parameter adds two vertices, so at fixed order there are $O(n^4)$ descriptions. Ambiguity of descriptions can only reduce the number of isomorphism classes. This proves the upper bound.

11.5 A matching five-parameter family

Define a rooted arm $A(a)$, $a \ge 0$, by taking a triangle xyz, adding two ears on xz, adding w on yz, and declaring the root to be (w, y), of degrees $(2, 3)$. Repeat a times: on root (ℓ, h) add t, then add z' on ht, retaining root (z', t). This is compact state 1, of weight two and order $6 + 2a$.

Identify the low root endpoints and the high root endpoints of two arms $A(a), A(b)$. The resulting cherry $B(a, b)$ has root degrees $(3, 5)$, state 15, weight four, and order $10 + 2(a + b)$. Starting with $B(c, d)$, add one vertex on its root edge and retain the new vertex followed by the old low endpoint as root. Its root degrees are $(2, 4)$, state 12, and weight three. Repeat r times: add an ear and a continuing core vertex on this root edge, retaining the new core vertex followed by the old low endpoint. Each such loop preserves state and weight and adds two vertices.

Finally identify this root crosswise with that of $B(a, b)$. Both final root endpoints have degree six and requirement six, so are interior. The resulting $F(a, b, c, d, r)$ satisfies

$$|V(F)| = 19 + 2(a + b + c + d + r), \qquad b(F) - i(F) = 4 + 3 - 2 = 5.$$

It is therefore extremal for every nonnegative choice of parameters.

Its canonical degree-two-pruned core incidence tree has two trivalent edge-nodes. The four pendant paths contain $2 + 2a, 2 + 2b, 2 + 2c, 2 + 2d$ triangle-nodes, in pairs at the two branches.

The path between the branches contains $r+1$ triangle-nodes. An isomorphism thus determines r and the unordered pair $\{\{a,b\},\{c,d\}\}$. At most eight ordered tuples can describe one isomorphism class. For sum S, there are $\binom{S+4}{4}$ tuples, yielding the stated lower bound. Combined with the upper bound, this proves $a_n = \Theta(n^4)$ through odd n.

The recovered incidence-tree lengths are graph invariants, as shown above. As a numerical check, all 792 tuples of sum at most seven, reaching order 33, pass fresh BFS boundary checks and exact parameter recovery. The subsequent section strengthens this result to an exact generating function and leading coefficient by resolving all isomorphism identifications.

11.6 Finite-state classification at every fixed excess

Theorem 11.2 (Effective fixed-excess grammar)**.** *For each fixed integer $e \geq 0$, the two-trees of order at least six with $2b - n - 4 = e$ are generated by an effectively constructible finite grammar, together with the explicit complete-core cases above. Put $C = 2e+6$, $L = 2e+2$, and $K = (C-2)^2(C-1)$. The grammar needs at most*

$$(e+1)(L-1)^2\frac{K(K+1)}{2} = O(e^9)$$

states.

Proof. Use (18) and the same rooted requirements $\max(A, B+t)$. A continuing endpoint has degree in $[2, C-1]$; cap A at $C+1$ and B at $C-1$. Since $t \geq 2$, both caps preserve all boundary comparisons. The endpoint triples thus have K possible numerical values. The inside root multiplicity and leaf count both lie in $[1, L-1]$. Sorting the two triples gives at most $(L-1)^2K(K+1)/2$ raw state/leaf pairs. Ear multiplicities are also bounded by degree. The four operations and final gluing remain exact, and cover every core of at most L leaves.

Every reachable component weight is bounded below. Indeed, fill its pending root with one outside triangle and no root ears. Previously forgotten statuses are unchanged because root multiplicity remains at least two. The completed two-tree has weight at least four at order at least six, and at least three at smaller orders. Its two former roots and new vertex contribute at most three. Thus all component weights are nonnegative, and state minima are attained integers. They can be computed by increasing leaf count: binary joins use smaller counts and unary extensions give a finite shortest-path problem without reachable negative cycles.

Reduced production costs are nonnegative. Every minimum completed gluing is an actual graph of weight at least four, so for target $e+4$, each subcomponent's slack is in $[0, e]$. Adjoining this budget coordinate, retaining productive states, and accepting exact total weight $e+4$ yields the claimed finite grammar and state bound. The complete-core cases have already been classified. □

This theorem does not assert polynomial unlabelled growth at every fixed excess. The deterministic cyclic structure required for the quartic statement was certified specifically for $e = 1$.

Formal verification scope. The Lean file `EvenEqualityAutomaton.lean` proves that every accepted run, of arbitrary length, has weight at least four and that every weight-four run uses only the four optimal transitions. The odd finite grammar is also accompanied by a universal derivation-weight and interior-degree certificate. The graph-to-state correspondence, core reduction, and counting argument are ordinary mathematical proofs supported by independent exact checkers; they are not claimed as end-to-end Lean formalizations.

12 Exact enumeration: a rational function and a period-eight formula

The finite grammar can be made unambiguous by using the canonical degree-two-pruned core. This gives an exact count, strengthening the quartic growth theorem.

Theorem 12.1 (Exact unlabelled equality count)**.** *There is one minimum-boundary two-tree at each of orders three, four, and five, namely K_3, $K_4 - e$, and the cone over P_4. There are two at order six, seven at order seven, and exactly three at every even order at least eight. For every odd $n \geq 9$,*

$$a_n = \frac{n^4 - 20n^3 + 239n^2 + c_r n + d_r}{192}, \qquad r = n \pmod 8,$$

where

r	1	3	5	7
c_r	-1126	-1090	-1126	-1090
d_r	2250	1962	2298	1914

In particular $a_n \sim n^4/192$ through odd n. With $t = x^2$, the exact generating function is

$$\sum_{\substack{n \geq 7 \\ n \text{ odd}}} a_n x^n = \frac{x^7 N(t)}{(1-t)^5(1+t)^2(1+t^2)}, \tag{20}$$

where

$$N(t) = 7 - 3t - 4t^2 + 8t^3 - t^4 + 4t^5 + 11t^6 - 8t^7 + t^8 + t^9.$$

12.1 Canonical arms and their generating functions

Use the state numbering of the compact odd grammar. For $j = 0, \ldots, 9$, let $F_j(x)$ count rooted component isomorphism classes in state j, with x marking vertices other than its two root endpoints. These are one-leaf components, so their pruned core is a triangle path. Their root endpoint triples are unequal and fix an orientation.

There is a unique decomposition of such a rooted graph. Retain the two marked root endpoints and remove every other degree-two vertex; this recovers its rooted core. Its root triangle either is the sole triangle, or has exactly one continuing core edge. The ear multiplicities and the continuing edge determine the last extension uniquely, including the predecessor orientation. Iterate towards the leaf. The only duplicate initial parameters are $(p, 0)$ and $(0, p)$ for $p = 1, 2, 3$, which describe the same canonically oriented cap and are counted once. Thus the unary transition equations are exact unlabelled equations, rather than equations for derivations. Their solutions are

j	$F_j(x)$	j	$F_j(x)$
0	x^2	5	$x^4(1 + 4t - 3t^2)/(1-t)^3$
1	$2x^4/(1-t)$	6	$2x^5/(1-t)^2$
2	$x^3(1 + 4t - 3t^2)/(1-t)^3$	7	x^4
3	x^3	8	$2x^4(1+t)/(1-t)$
4	$x^3(1+t)/(1-t)$	9	$x^4(1+t^2)/(1-t)^2$

These identities follow from all one-leaf rules with the three initial seeds $F_0 = x^2$, an added x^3 in F_4, and an added x^4 in F_9. Every extension increases order, so the solution as formal power series is unique.

Define

$$A = F_0 + F_1 = \frac{x^2(1+t)}{1-t}, \qquad B = F_3 + F_4 + F_6 = \frac{2x^3}{(1-t)^2}.$$

The A-arms have weight two and root degrees $(2,3)$; the B-arms have weight three and root degrees $(2,4)$. For an unlabelled class C, unordered pairs have generating function

$$\mathrm{Pair}(C)(x) = \frac{C(x)^2 + C(x^2)}{2}.$$

Set

$$M = \mathrm{Pair}(A) = \frac{x^4(1+t+2t^2)}{(1-t)^2(1+t)}.$$

12.2 One or two branches

If the core has one branch, its unique trivalent edge-node canonically roots the graph. Each arm has low root degree two and high root degree at least three. One root ear would make the sum of final root degrees at least $3 \cdot 5 - 4 + 2 = 13$, so there are no root ears. No endpoint can be high in all three arms, which would give degree at least seven. Each therefore has requirement at least six. The degree sum is at least eleven; degrees five and six would give one boundary and one interior root, and weight at least six. Hence both root endpoints are interior of degree six. The three arm weights must be $2, 2, 3$. The endpoint degree equations force the weight-three arm to have root degrees $(2,4)$, and the two weight-two arms to have their degree-three sides aligned with its degree-two side. Thus there are two A-arms and one B-arm. The unique B-arm fixes the root ordering, and the A-arms form an unordered pair. Every such combination has requirement six at both final endpoints. Therefore the exact three-leaf contribution is

$$U_3 = x^2 MB = \frac{2x^9(1+t+2t^2)}{(1-t)^4(1+t)}. \tag{21}$$

With two branches, the same root equations force two A-arms at each branch and state 12 for the component leading to the other branch. The only entries into state 12 are an extension of states 14 or 15, which are a pair of A-arms, and its degree-six self-loop. Hence the graph is determined by an unordered pair of unordered pairs of A-arms, and a central length $r \geq 0$. The core canonically recovers all these data. Reversing the central strip exchanges its two branches: the central core is the square of a path, with one ear on each strictly internal consecutive edge, and the higher arm-degree endpoint at the outer end of either terminal edge. This reversal preserves the endpoint roles for every r. Consequently the exact four-leaf contribution is

$$U_4 = \frac{x^3}{1-x^2}\,\mathrm{Pair}(M) = \frac{x^{11}(1+4t^2+2t^3+6t^4+3t^6)}{(1-t)^5(1+t)^2(1+t^2)}. \tag{22}$$

The canonical decomposition proves that repeated or equal arms are counted correctly; there is no assumption that grammar derivations are unambiguous.

12.3 Path cores and reflection

For a path core, mark one of its two end triangles. Its core ear and opposite internal edge are intrinsic. Exactly one closed edge of this end triangle has ears, so this fixes the root endpoint order. The end component is a literal initial cap in state 0,4, or9. The terminal table with this cap first and its orientation zero gives the end-rooted function

$$\begin{aligned} E &= x^4(F_2 + 2F_3 + 2F_4 + F_6) \\ &\quad + x^5(3F_0 + 3F_1 + F_5 + F_7 + F_8 + F_9) \\ &\quad + x^6(2F_3 + 2F_4 + F_6) \\ &= \frac{4x^7(2+t-3t^2+t^3)}{(1-t)^3}. \end{aligned}$$

Selecting the other end is an involution. Its fixed classes are exactly graphs admitting an end-exchanging automorphism. If their generating function is S, Burnside's elementary two-element formula gives $U_2 = (E + S)/2$.

The center of the core incidence path is an edge-node or a triangle-node. Its symmetry cases are exhaustive as follows; q counts ears on the central edge, or on each of the two internal edges at a central triangle.

center	half states	matching	q
edge	$0, 1$	same endpoint order	1
edge	$3, 4, 6$	opposite endpoint order	1
triangle	$0, 1$	shared corner high	0
triangle	$3, 4$	shared corner low	0
triangle	$3, 4, 6$	shared corner low	1

Here "low" and "high" refer to the half's two root degrees. At an edge center the weight equation is $2w + q + s_u + s_v = 5$, so parity forces $q = 1$; the root degree and requirement tests give the first two rows. At a triangle center, let u be the shared corner and v, w the exchanged corners. Initially allow r ears on the third edge. Then $D_u = 2d_u + 2q$, $D_v = D_w = d_v + q + 1 + r$, and the weight is $2w + 2q + r + s_u + 2s_v$. The degree bound gives $q \le 1$. If $r > 0, q = 1$, the shared corner would have degree six and requirement at least eight, impossible. If $r > 0, q = 0$ and the shared corner is high, the other two corners are boundary and the weight is at least $4 + r + 1 \ge 6$. If the shared corner is low, it is boundary. Half high degree three then gives weight at least eight; high degree four allows only $r = 1$ and gives weight at least six; larger degrees are impossible. Thus $r = 0$. The degrees reduce to $D_u = 2d_u + 2q$ and $D_v = D_w = d_v + q + 1$, and the weight equation is $2w + 2q + s_u + 2s_v = 5$. If u is low, its requirement is $5 + q$, and that at v is $\max(A_v, 5 + q)$. If u is high, the degree-six bound forces high degree three and $q = 0$. These conditions give exactly the remaining three rows.

Isomorphic rooted halves always give the required reflection; use an isomorphism in one direction and its inverse in the other. Conversely every end exchange identifies the halves in one listed way. Unequal root summaries preclude another root-swapping identification within a half. The center type and its ear multiplicities are intrinsic, so the cases are disjoint. Doubling the half replaces x by x^2, giving

$$\begin{aligned} S_{\text{edge}} &= x^3(F_0 + F_1 + F_3 + F_4 + F_6)(x^2), \\ S_{\text{triangle}} &= x^3(F_0 + F_1 + F_3 + F_4)(x^2) + x^5(F_3 + F_4 + F_6)(x^2), \\ S &= \frac{2x^7(1 + t - t^3)}{(1 - t)^2(1 + t)}. \end{aligned}$$

Thus

$$U_2 = \frac{x^7(5 + 6t - 5t^2 - 5t^3 + 3t^4)}{(1 - t)^3(1 + t)}. \tag{23}$$

The path-core count is five at order seven and, for odd $n \ge 9$, equals $(n^2 - 25)/4 + \lfloor (n + 1)/4 \rfloor$.

12.4 Exact algebra and independent checks

The complete-core exceptions contribute $2x^7$. The other classes have respectively two, three, and four core leaves, so are disjoint. Adding $2x^7 + U_2 + U_3 + U_4$ yields (20).

To certify the quasipolynomial without extrapolation, put $m = (n - 7)/2 = 4k + r$. Its four Newton coefficient rows, indexed by $r = 0, 1, 2, 3$, are

0	8	109	502	896	512
1	18	177	678	1024	512
2	36	280	886	1152	512
3	66	420	1126	1280	512

The corresponding polynomial is $P_r(k) = \sum_{j=0}^{4} c_{rj}\binom{k}{j}$; subtract one only at $m = 0$. Since $\sum_{k\geq 0}\binom{k}{j}z^k = z^j/(1-z)^{j+1}$, these rows give a rational function with denominator $(1-t^4)^5$. Cross-multiplication with (20) is an exact integer polynomial identity. Expanding the four polynomials in n proves the displayed residue formula for every odd $n \geq 9$, with $a_7 = 7$ as its sole initial correction.

The standard-library checker `exact_odd_count.py` verifies every one-leaf recurrence, terminal sum, symmetry formula, unordered-pair identity, and the exact polynomial certificate. A separate program of the earlier session generates the complete grammar by increasing order and canonicalizes graphs using recursive ordered-edge incidence codes. Equality of these codes is exact equality of recursive tuples; rerooting at all edges and both orders gives unlabelled isomorphism. It reproduces every individual nauty class through order 13 and all stratified counts through order 35, where there are 4686 extremizers. These experiments check the all-order canonical proof rather than replace it. The counting correspondence is not represented as an end-to-end Lean formalization. The exact integer-polynomial identities for the summed numerator and Newton coefficient formula are separately kernel-checked in `OddCountingIdentities.lean`.

13 Rational enumeration at every fixed excess

The fixed-excess grammar can be strengthened to an effective rational enumeration theorem. The proof counts canonical colored incidence trees, rather than ambiguous grammar derivations.

Theorem 13.1 (Rational unlabelled enumeration at fixed excess)**.** *Fix an integer $e \geq 0$, and let $\mathcal{C}_e$ be the isomorphism classes of two-trees G of order at least six with $2|\partial G| - |V(G)| - 4 = e$. Then*

$$F_e(z) = \sum_{[G]\in\mathcal{C}_e} z^{|V(G)|}$$

is an effectively computable rational function. In particular, for each fixed e, its exact unlabelled counts eventually satisfy a constant-coefficient linear recurrence.

The restriction to orders at least six avoids the exceptional small orders in the sharp bound. Extending a fixed-excess count to all orders changes its generating function by only a finite polynomial. No practical complexity bound is asserted for the general algorithm.

13.1 Finite bounds and an intrinsic pending-root weight

Put

$$C = 2e + 6, \qquad D = \max(4, 2e + 3), \qquad L = 2e + 2.$$

The structural defect theorem bounds interior degrees by C and boundary degrees by D. Remove all degree-two graph vertices to obtain the canonical core H. If H is noncomplete, it is a two-tree with at most L ears. Its internal triangle–edge incidence tree, obtained by deleting private edge-nodes, has at most L leaves. Every removed vertex is an ear on a core edge f; write its multiplicity as q_f. Since each core endpoint has degree at least two in H,

$$0 \leq q_f \leq C - 2.$$

The complete cores K_2, K_3 have the already classified orders $e + 4$ or $e + 6$ and bounded ear multiplicities, so contribute a finite polynomial. We treat noncomplete cores below.

Cut at an internal core edge uv, with at least one core triangle on either side, and defer all ears on that root edge. Exclude u, v from the component weight W. Every other vertex receives its boundary/interior sign under the *pending-root convention* that the final multiplicity of uv is

at least two. This is not the raw boundary weight of the isolated component. Indeed, adding the first outside triangle can activate a local condition at a page vertex. Under the pending convention that condition is already active: the local formula

$$\beta(u) = R_u - \deg u, \quad R_u = \max\{2 + t_{uv},\ 2 + t_{ua} + t_{ub} : uab \text{ a triangle},\ t_{ab} \geq 2\}$$

depends on the root edge, for a nonroot vertex, only through the threshold $t_{uv} \geq 2$. All incident edges and degrees of that nonroot vertex are closed. Thus W is intrinsic and independent of the chosen completion with an active root edge.

Every such component of a graph in $\mathcal{C}_e$ satisfies

$$0 \leq W \leq e + 6. \tag{24}$$

To prove the lower bound, add one vertex on the root edge. Its multiplicity is now at least two, preserving exactly the pending nonroot signs just defined. Every two-tree of order at least three has total boundary-minus-interior weight at least three, by the sharp theorem and the direct small-order check. This completion has at least four vertices, so a convention also admitting K_2 is irrelevant. Its two former roots and new degree-two vertex contribute at most three, proving $W \geq 0$.

For the upper bound, let W' be the pending nonroot weight of the complementary component, and q the root ear multiplicity. With root signs $s_u, s_v \in \{-1, 1\}$, the final graph gives

$$e + 4 = W + W' + q + s_u + s_v.$$

Since $W' \geq 0$, $q \geq 0$, and $s_u + s_v \geq -2$, the upper bound in (24) follows. The argument also applies to any nonempty proper group of pages at an internal edge.

13.2 An exact finite-alphabet incidence encoding

A two-tree has a proper three-coloring unique up to one global permutation: name the colors on an initial triangle, after which each added vertex is forced to use its two parents' missing color. For a named coloring of H, color every edge-node of its full triangle–edge incidence tree by the missing endpoint color. The three edge-nodes at a triangle have different colors. This colored tree reconstructs the colored graph uniquely. Root at a triangle, name its three vertices by color, and traverse the tree; every new triangle shares a known edge and introduces one new vertex of the missing color, while its two other edge colors determine the incidences. Conversely this procedure reconstructs a two-tree from every locally valid such colored incidence tree.

Use the internal incidence tree T, obtained by deleting all edge-nodes lying in just one triangle. Give its nodes finite labels:

- an internal edge-node carries its missing color and q_f;
- a triangle-node records the subset of edge colors remaining internal, and the ear multiplicity for each missing private edge color.

Its neighboring edge colors must be precisely that recorded subset. All deleted private edges can be restored uniquely. Thus these labels reconstruct H and then G. Ears on one edge are represented by their multiplicity, so their permutations are already removed.

The alphabet is finite. Triangle degrees are at most three, and edge-node degrees are at most L, since T has at most L leaves. All leaves are triangle-nodes. Define a node weight by

$$\omega(v) = \begin{cases} 1 + \sum q_f, & v \text{ a triangle-node, over its private edges,} \\ q_f, & v \text{ an internal edge-node.} \end{cases}$$

Then

$$|V(G)| = 2 + \sum_{v \in T} \omega(v). \tag{25}$$

A core with t triangles has $t + 2$ vertices, and every ear has been counted once. All weights are nonnegative and triangle weights are positive. Color permutations preserve them. Colored decorated tree isomorphism is exactly colored graph isomorphism; forgetting colors adds precisely the one global S_3 action, with no separate gluing gauge at each edge.

13.3 A deterministic recognizer on the canonical tree

Root T at a designated leaf triangle. At an internal edge cut, order the two graph endpoints by their named colors and retain: the number of inside root triangles; both current endpoint degrees; both requirements $\max(A, B + t)$, with A capped at $C + 1$ and B at $C - 1$; the inside leaf count; actual pending weight W in $\{0, \ldots, e + 6\}$; and the finite deferred-ear and color data. Continuing degrees are at most $C - 1$. The requirement caps preserve every final comparison at degree at most C, since $t \geq 2$. All coordinates are finite; include a rejecting state.

There are canonical deterministic transitions. A triangle has zero, one, or two active child core edges, whose ports are distinguished by color. The local boundary formula computes the affine root requirements and the final sign and degree of its third vertex; labels specify all ear multiplicities. Add that sign and the ears to W. At an edge-node combine *one unordered star of all child triangle branches at once*: root degrees add with subtraction for the repeated root neighbor, requirements combine by maxima, and triangle counts, leaves, and nonroot weights add. This transition is symmetric in a bounded multiset of child states, retaining the full multiplicity of repeated branches. No binary-join history is counted. The affine constants can be defined intrinsically as maxima of constant and root-linear local contributions, rather than by a choice among equivalent formulas.

Require each finalized retained core vertex to have degree at least three and satisfy the appropriate boundary/interior cap. Reject if any state bound is exceeded. At the root leaf triangle, close the remaining edge, account for its ears and the two private edges, finalize all remaining vertices, and accept exactly at total weight $e + 4$. The root core vertices must also finish with degree at least three.

Every member of $\mathcal{C}_e$ passes under any chosen leaf root, by the proved bounds and exact local formula. Conversely an accepted decorated tree reconstructs a two-tree whose specified core vertices all have degree at least three and whose attached ears have degree two. Its specified core is therefore the canonical pruned core. All signs are correct and total weight $e + 4$ is exactly the desired excess. A noncomplete pruned core with all ears retained above degree two gives order at least six. Thus the deterministic recognizer accepts exactly the required graph property. Intermediate states may depend on the designated root, but acceptance does not.

13.4 Finite skeletons and a disjoint language partition

Suppress all degree-two *incidence nodes* of T. This is distinct from removing degree-two graph vertices. The resulting skeleton K has at most L leaves and $L - 2$ branch nodes, hence at most $2L - 2$ nodes. Choose one named representative of each of the finitely many abstract skeletons, a designated leaf, and an orientation of each edge. A decorated core with this skeleton is specified by finite labels on the skeleton nodes and one word on each oriented edge, listing its suppressed degree-two node labels. Local type and color compatibility are finite conditions. Chain reversal is literal word reversal and color action is letterwise.

Along a chain, rooted towards the designated leaf, the recognizer applies unary maps on its finite state set. Adjoin rejection to make every map total. A chain word has one exact composite map

in a finite transformation monoid. For any fixed map, its words form a regular language, and these languages are disjoint. Endpoint compatibility can be incorporated in the finite machine. Once all skeleton labels and composite chain maps are fixed, acceptance is decided by the finite remaining computation at skeleton nodes.

The valid configurations on a named skeleton are consequently a *disjoint finite union of products of regular chain languages*, with fixed skeleton labels. This exact-map partition avoids counting a graph once per derivation or state annotation.

The word weight is the sum of the node weights in (25). Edge letters can have weight zero, but types alternate and triangle letters have positive weight; a word of length h has weight at least $\lfloor h/2 \rfloor$. Thus there are finitely many words of each weight. On a productive recognizing automaton no zero-weight directed cycle is possible, and the weighted transfer-matrix expression is a well-defined rational formal power series.

13.5 Fixed words under twisted reversal

We need a finite-language fact that includes reflections. Let $\mathcal{L}$ be a regular language on a finite nonnegatively weighted alphabet, with finitely many realizations at each weight, and let ρ be a weight-preserving permutation of its letters. The generating functions for

$$\{w \in \mathcal{L} : w = \rho(w)\}, \qquad \{w \in \mathcal{L} : w = \rho(\operatorname{rev} w)\}$$

are rational. The first language restricts to letters fixed by ρ. For the second, applying the equation twice restricts all letters to $\mathrm{Fix}(\rho^2)$, on which ρ is an involution. Each fixed word is uniquely

$$u\rho(\operatorname{rev} u)quad\text{or} \quad ua\rho(\operatorname{rev} u), \qquad \rho(a) = a.$$

Let a deterministic automaton for $\mathcal{L}$ have initial state q_0, accepting set F, and letter transitions δ_b. While reading u, track

$$\big(\delta(q_0, u), T_{\rho(\operatorname{rev} u)}\big),$$

where T_v is the full state transformation for word v. This is a finite state. Appending a letter b updates it by

$$(p, T) \longmapsto \big(\delta_b(p), T \circ \delta_{\rho(b)}\big). \tag{26}$$

For even length accept if $T(p) \in F$. For an odd center a, accept if $T(\delta_a(p)) \in F$. Therefore the half-words form regular languages. Give a half-letter twice its original weight and add the central weight where present. This proves rationality, counting each fixed word once by its unique half-word. The alternating-type weight bound also precludes productive zero-weight cycles here. Intersections and inverse images under letter permutations or reversal preserve regularity, so the same argument applies with finitely many additional chain constraints.

13.6 The full isomorphism quotient and Burnside count

For a fixed named skeleton the finite group is

$$\Gamma_K = \mathrm{Aut}(K) \times S_3.$$

The first factor permutes nodes and chains and reverses their words when it reverses the selected orientation; the second acts globally on all color labels. The actions commute and preserve weights. Their orbits are exactly the uncolored graph isomorphism classes. An isomorphism preserves the canonical core, its incidence tree, and its suppression. On a chain it is determined by the endpoints and hence preserves or reverses the word; there are no translations or extra local gauges. The colored incidence encoding is bijective, and forgetting the unique named coloring

adds precisely S_3. Conversely every such group identification reconstructs an isomorphism of the graphs. Ear permutations are already represented by multiplicity counts.

For $g \in \Gamma_K$, choose an assignment of skeleton labels fixed by g, and one accepting product of the disjoint classes defined by chain maps. On an orbit of m chains, all words are determined by one representative w through specified color permutations and reversals. Pull the regular conditions on the other chains back to w and intersect them. Going around the orbit imposes exactly

$$w = \rho(w) \quad \text{or} \quad w = \rho(\operatorname{rev} w)$$

for an accumulated color permutation ρ. The preceding half-word construction gives a rational representative series. All m copies have the same weight, so use the substitution $z \mapsto z^m$. Different chain orbits are independent once their finite maps and labels are fixed, so their series multiply. Summing over the finite disjoint products gives a rational fixed configuration series for g.

The designated recognizer root need not be fixed by g: acceptance is the intrinsic graph property. Nor must the exact-map partition be group-invariant; intersecting each disjoint product with the fixed-point conditions counts every fixed configuration once. Burnside averaging

$$\frac{1}{|\Gamma_K|} \sum_{g \in \Gamma_K} (\text{valid configuration series fixed by } g)$$

therefore counts exactly the unlabelled graphs of this skeleton and is rational. Include the finite skeleton-label monomials and the global factor z^2 from (25), sum over the finitely many skeletons, and add the complete-core polynomial. This proves Theorem 13.1.

Effectivity and scope. All objects in this construction are finite and enumerable: bounded labels and states, canonical incidence transitions, skeletons and their automorphism groups, exact chain transformation maps, regular-language intersections, and the half-word automata (26). Weighted transfer matrices and finite-group averages use exact rational arithmetic. This gives a terminating construction for any specified e, although its transformation monoids can be very large. Rationality implies eventual constant-coefficient recurrences; it does not imply polynomial growth, quasipolynomial coefficients, or only roots of unity as poles. Those stronger properties were proved explicitly for $e = 0, 1$. The present structural/automata proof has passed independent mathematical audit; it is not an end-to-end Lean formalization.

14 A further general exclusion: Cartesian products

Proposition 14.1. *For connected graphs H, J,*

$$\beta_{H \square J}(h, j) = \beta_H(h) + \beta_J(j).$$

In particular every graph with five boundary vertices is Cartesian-prime: it has no decomposition with both factors having at least two vertices.

Proof. Product distance is the sum of factor distances. Splitting the neighbors into the two changing coordinates gives $\beta_{H \square J}((h, j), (u, v)) = \beta_H(h, u) + \beta_J(j, v)$. The two targets are independent, so maxima add. Thus $\partial H \times \partial J \subseteq \partial(H \square J)$. If the product boundary has five vertices and both factors are nontrivial, then $b(H), b(J) \geq 2$ and $b(H)b(J) \leq 5$, forcing both to equal two. Both factors are paths. Directly summing the path stability values gives $b(P_m \square P_n) = 2m + 2n - 4$ for $m, n \geq 2$, which is even. □

The product identity is elementary and is recorded as a secondary exclusion, not as the main contribution.

15 Sharp local metric reduction for every boundary size

The five-point cone argument above proves the unrestricted classification. We now give a separate metric theorem that applies to any number of designated points and any radius. It replaces metric-cone enumeration for a *single rooted local problem*; compatibility between different roots remains a separate requirement.

Theorem 15.1 (Sharp rooted metric representative)**.** *Let M be an integer metric with distinguished point o, and let $R \geq 0$ be an integer. For each point i put*

$$D_i = M(o,i), \qquad H_{ij} = D_i + D_j - M(i,j), \qquad T = 2R+1.$$

Set $D'_i = D_i$ when $D_i \leq R$ and $D'_i = T$ otherwise. For distinct points set $H'_{ij} = \min(H_{ij}, T)$ and

$$M'(i,j) = D'_i + D'_j - H'_{ij}, \qquad M'(i,i) = 0.$$

Then M' is an integer metric on the same points, with root distances D'_i. It preserves the clipped rooted signature

$$\big(\min(D_i, R+1),\ \min(H_{ij}, 2R+1)\big)_{i,j},$$

and satisfies

$$\max_i M'(o,i) \leq 2R+1, \qquad \operatorname{diam}(M') \leq 4R+2.$$

The transformation is idempotent. Both constants are best possible: four points force the root-distance bound and five points force the diameter bound.

Proof. The metric inequalities are equivalent to the relevant relations

$$0 \leq H_{ij} \leq 2\min(D_i, D_j), \quad H_{ij} < D_i + D_j \ (i \neq j), \quad H_{ij} + H_{jk} - H_{ik} \leq 2D_j.$$

Symmetry is immediate. If both D_i, D_j are at most R, their mutual distance is unchanged. If both exceed R, their new distance is at least T. In the mixed case their new distance is at least $T - D_j \geq R+1$, where D_j is the smaller root distance. Thus distinct points remain distinct.

The triangle inequality through j becomes $H'_{ij} + H'_{jk} - H'_{ik} \leq 2D'_j$. If $D_j > R$, the left side is at most $2T = 2D'_j$. If $D_j \leq R$, its two incident H values are unchanged. When $H_{ik} \leq T$ the old triangle inequality applies. Otherwise

$$H'_{ij} + H'_{jk} - H'_{ik} \leq 4D_j - T \leq 2D_j.$$

The root distances, signature identities, bounds, and idempotence now follow from the definitions.

For root-bound sharpness take points o, i, j, k with

$$M(o,i) = M(o,k) = M(i,j) = M(j,k) = R+1, \quad M(o,j) = T, \quad M(i,k) = 2R+2.$$

These distances form a metric. Its products satisfy $H_{ij} = H_{jk} = T$ and $H_{ik} = 0$. Any representative with the same clipped signature therefore has $2D'_j \geq T + T - 0$, so $D'_j \geq T$.

For diameter sharpness take four cyclically ordered points and a root. Give each root distance and each cyclic-edge distance the value T, and each opposite-pair distance $2T$. This is a metric. Products are T on cyclic edges and zero on opposite pairs. For each cyclic point the preceding triangle inequality forces root distance at least T. Each opposite pair still has product zero, so its distance is at least $2T = 4R+2$. □

Theorem 15.2 (Preservation of the complete local frame)**.** *At radius R, the preceding transformation preserves every positive profile, every designated point in the ball, sup-norm adjacency, and all signed neighbor-score coefficients toward designated points, in normalized coordinates.*

Proof. At root-distance $1 \le r \le R$, write $f_o = r$ and $f_i = D_i + \epsilon_i$ with $-r \le \epsilon_i \le r$. The positive-frame conditions become

$$D_i + \epsilon_i > 0, \qquad \epsilon_i + \epsilon_j \ge -H_{ij}, \qquad \exists i : \epsilon_i = -r,$$

and, for every i, either $\epsilon_i = -r$ or some $j \ne i$ satisfies $\epsilon_i + \epsilon_j = -H_{ij}$. Positivity is automatic whenever $D_i > R$. When $H_{ij} > 2R$, its inequality is automatic and cannot be tight. All other quantities in these tests are unchanged, giving the required bijection.

A designated point c lies in the ball exactly when $D_c \le R$. Its normalized coordinates are $r = D_c$, $\epsilon_c = -D_c$, and $\epsilon_i = D_c - H_{ci}$ for $i \ne c$. Since $H_{ci} \le 2D_c \le 2R$, these coordinates are unchanged. Finally, keeping every offset keeps every coordinate difference. This preserves sup-norm distances, adjacency, and all signed coefficients in the local harmonic inequalities. □

For k designated points, a direct finite search now needs at most

$$(R+1)^{k-1}(2R+2)^{\binom{k-1}{2}}$$

rooted signatures before the metric conditions and symmetries are imposed. This is a uniform finite reduction for all k, R, with no classification of metric cones as an input. For $k = 5, R = 2$, direct generation gives 13,022 canonical metric types. The stronger experimental local model leaves 1,226 feasible types. These latter computations are discovery checks, not a replacement for the independently certified all-root proof in the main classification.

A necessary limitation. The theorem preserves the data at the chosen root only. At $R = 2$, a three-point metric with $D_a = 3$, $D_b = 4$, and $M(a, b) = 1$ becomes $D'_a = D'_b = 5$, $M'(a, b) = 5$. The root-o signature is preserved while the root-a signature changes. Thus one cannot obtain a simultaneous global cap merely by applying the transformation at different roots.

Formal and independent verification. `RootedMetricTruncation.lean` proves the actual metric axioms on an arbitrary index type, the signatures and bounds, idempotence, full local positive-profile invariance, marked-point offsets, coordinate differences, and both sharpness examples. All sixteen principal axiom audits passed in Lean 4.35.0; the file contains forty theorem declarations and uses only the standard logical foundations. A separate exact checker examined 6,321 four-point metrics, 102,000 rooted instances, 96 large-weight adversarial metrics, 864 full-profile comparisons, and the sharpness families for $0 \le R \le 20$. The finite tests corroborate the universal proof; their range is not the theorem's range.

16 An exact topological boundary theorem and the six-boundary sequel

Theorem 16.1 (Bipartite subcubic boundary)**.** *For every finite simple connected bipartite graph G of maximum degree at most three and order at least two,*

$$\partial G = \{v : \deg(v) = 1\} \ \cup \ \{v : v \text{ lies on a cycle}\}.$$

Thus its harmonic boundary can be computed in linear time from a bridge search, without computing all-pairs distances.

Proof. Suppose $u \notin \partial G$ and $\deg(u) = 2$ or 3. In a bipartite graph, for any target z, each neighbor of u is at distance exactly $d(u, z) - 1$ or $d(u, z) + 1$ from z. If $z \ne u$, a shortest path provides

at least one closer neighbor. Two closer neighbors would give a positive score when $\deg(u) \le 3$. Hence there is exactly one closer neighbor.

Partition $V(G) \setminus \{u\}$ according to that unique first neighbor. There cannot be an edge between two different parts. Indeed, its endpoints have root distances differing by one, by bipartiteness. A shortest path to the nearer endpoint followed by that edge would give the farther endpoint another first neighbor. Therefore different neighbors of u lie in different components of $G - u$. In particular, u lies on no cycle.

Conversely, if u lies on no cycle, every incident edge is a bridge. For each target other than u, exactly one neighbor is closer and all other neighbors are farther. Thus $\beta(u) = 2 - \deg(u)$, and such a vertex is boundary exactly when it is a leaf. Leaves themselves have stability one. □

Both hypotheses have sharp local counterexamples: the nonbipartite theta graph with path lengths $1, 4, 4$ has two interior degree-three vertices on cycles, while the center of the bipartite 3×3 grid is an interior degree-four vertex on cycles.

Theorem 16.2 (Exact reduced-order identity)**.** *Let K be obtained from a graph in Theorem 16.1 by suppressing degree-two vertices lying on no cycle. Let $B_1, \dots, B_c$ be its nonbridge blocks, and let a be the total number of bridge incidences at their vertices. With $b = |\partial K|$,*

$$|V(K)| = 2b - 2 - \sum_{j=1}^{c} (|V(B_j)| - 2) - a \le 2b - 2.$$

Equality holds exactly when K is a tree all of whose nonleaves have degree three. The bound is attained for every $b \ge 2$.

Proof. Two distinct nonbridge blocks cannot share a vertex: that vertex would have degree at least four. Their vertices are therefore disjoint. Put $S = \sum_j |V(B_j)|$, let ℓ count leaves, and let t count the remaining vertices of K, all of degree three. The preceding theorem gives $b = S + \ell$. Contract each nonbridge block to one node. The resulting tree has ℓ degree-one vertices, t degree-three vertices, and c block nodes whose total degrees are a. The tree degree identity gives $\ell = t + 2 + a - 2c$. Substituting in $|V(K)| = S + \ell + t$ proves the formula. Every nonbridge block has at least four vertices, so any such block makes the inequality strict. The stated trees give equality. □

Corollary 16.3 (Constructive classification at every boundary size)**.** *Fix $b \ge 2$. Choose vertex-disjoint two-connected bipartite subcubic blocks with total order $S \le b$. Join them in a tree whose other nodes are $b - S$ leaves and degree-three branch vertices. At each block, attach the incident bridges injectively to distinct degree-two vertices of the block. Retain precisely the choices making a connected graph of order at least two. Subdivide any bridges arbitrarily. These constructions give exactly all connected bipartite subcubic graphs with boundary size b. There are finitely many reduced kernels, all of order at most $2b - 2$, and at most $\lfloor b/4 \rfloor$ nonbridge blocks.*

The necessity follows from the unique block decomposition and Theorem 16.1; sufficiency follows from the same boundary formula. This supplies an all-order classification for every boundary size within this infinite class.

Theorem 16.4 (Complete six-boundary bipartite subcubic classification)**.** *Every connected bipartite subcubic graph with six harmonic boundary vertices is obtained by arbitrary subdivisions of bridges of exactly one of the ten kernels in the following table. Conversely every listed kernel and every such subdivision has boundary six.*

Kernel description	*Number*	*Order*
Trivalent tree with six leaves; the four internal vertices induce P_4 *or* $K_{1,3}$	*2*	*10*
C_4 *with one leaf at each of two adjacent or opposite vertices*	*2*	*6*
C_4 *joined by one bridge to a new degree-three vertex with two leaf neighbors*	*1*	*7*
$K_{2,3}$ *with one leaf attached at a degree-two vertex*	*1*	*6*
$K_{3,3}$ *with a matching of size* $0, 1, 2,$ *or* 3 *deleted*	*4*	*6*

Proof. All vertices in nonbridge blocks count toward the six boundary vertices. Since each such block has order at least four and blocks are disjoint, there is at most one. If none exists, the reduced graph is one of the two six-leaf trivalent trees. If a block has order four, it is C_4 and exactly two leaves remain. The tree identity gives either two direct leaf attachments, adjacent or opposite, or a single attached two-leaf branch. If the block has order five, it is $K_{2,3}$ and the remaining leaf must attach at one of its symmetric degree-two vertices. If it has order six, there are no outside vertices. Its bipartition must have sizes three and three, and minimum degree two means its missing edges form a matching in $K_{3,3}$. This gives the four final entries. Bridge subdivision preserves both the stated topology and boundary. □

16.1 Removing bipartiteness at boundary six

The following section proves an all-orders reduction for nonbipartite subcubic graphs as well, and then classifies the entire connected subcubic six-boundary class. The bipartite theorem above remains stronger in another direction: it applies to every boundary size and has a full graph-level Lean proof.

For comparison, unrestricted two-connected searches at orders eight, nine, and ten find respectively twelve, two, and one six-boundary graphs. The last layer contains 9,743,542 graphs. This unrestricted census is only finite evidence: the higher-degree six-boundary core problem remains open in this work. These searches were made in the earlier session and are not part of the package.

17 Complete classification of subcubic six-boundary graphs

Theorem 17.1 (Six boundary vertices, maximum degree three)**.** *There are exactly twelve two-connected subcubic graphs with six harmonic boundary vertices: the ten two-connected subcubic graphs of order six, the theta graph with path lengths* $1, 4, 4$*, and the subdivision of* K_4 *whose three edges in one triangle have length three and whose remaining three edges have length one. In particular, their maximum order is ten.*

Every connected subcubic graph with six harmonic boundary vertices is obtained by arbitrary independent subdivisions of bridges of exactly one of 201 reduced kernels. Their maximum order is eighteen, attained by exactly two kernels. Both consist of four triangles joined by bridges, with six additional leaves; the four triangle nodes induce P_4 *or* $K_{1,3}$ *in the contracted tree.*

The new step is an all-orders structural proof for the interior of a subcubic core. It turns a finite census into an exhaustive classification. The subsequent block closure is independently certified without orbit pruning.

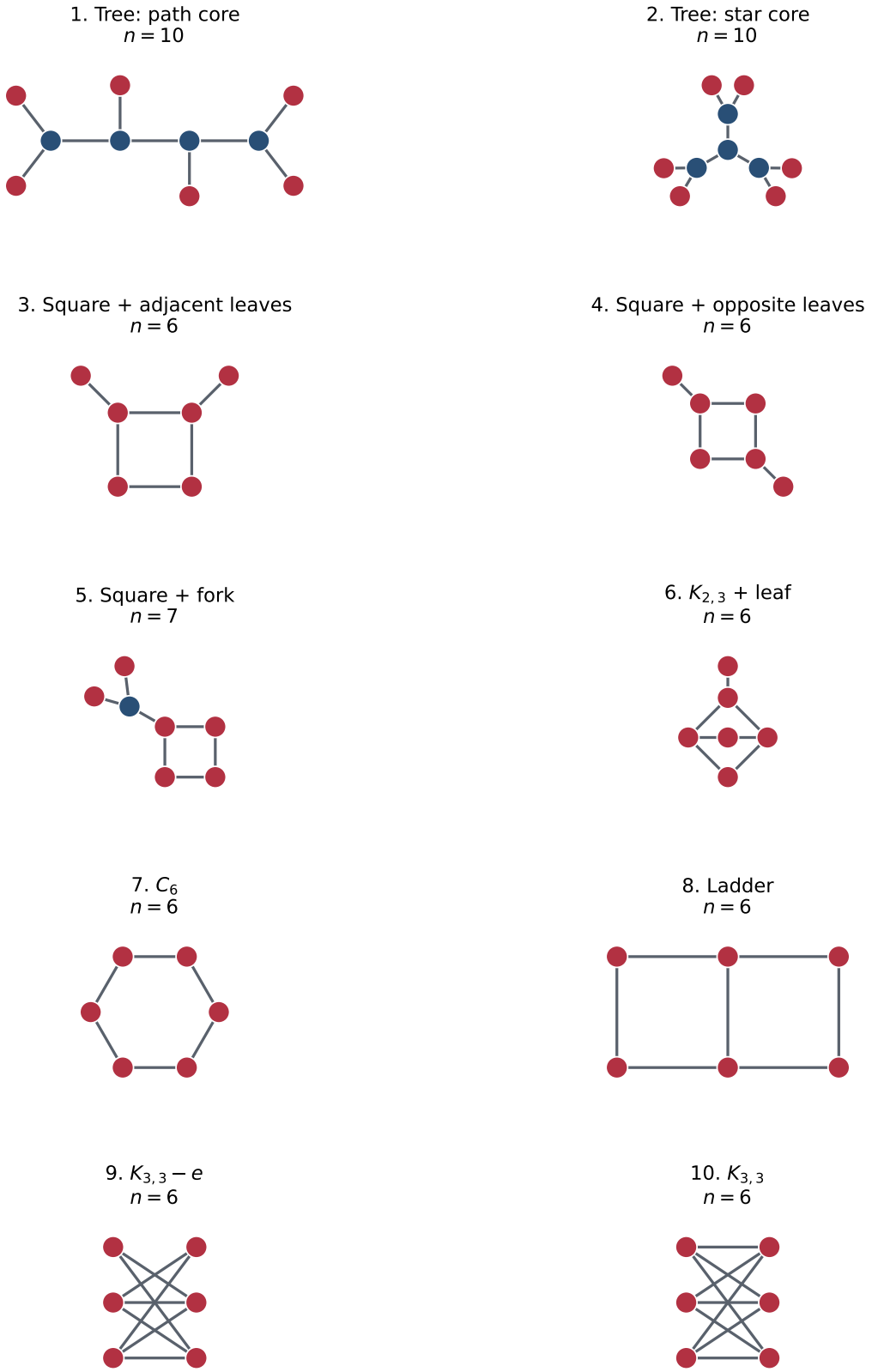


Figure 4: The ten bipartite subcubic six-boundary kernels. Boundary vertices are red and interior vertices blue. Only bridges may be subdivided in this construction; all such subdivisions are allowed. Unmarked edge crossings are not vertices.

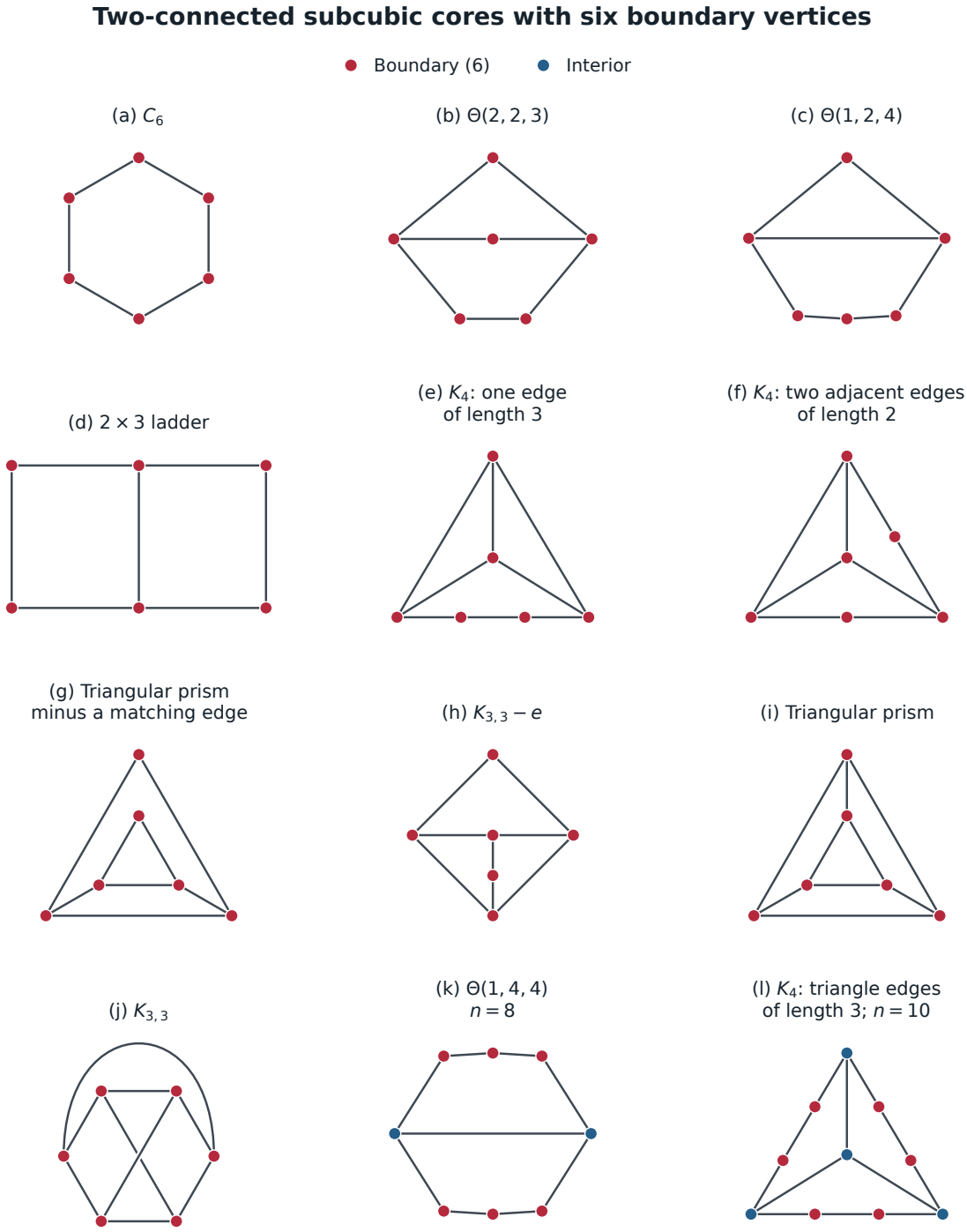


Figure 5: The complete twelve-core subcubic six-boundary palette. The first ten graphs have order six and all vertices boundary. The last two have orders eight and ten. Red marks boundary vertices; blue marks interior vertices. Every model is matched isomorphically to the exact catalogue and checked by fresh integer shortest-distance calculations.

17.1 The exact equations at an interior vertex

Until the block-assembly step, let G be two-connected and subcubic, $B = \partial G$ with $|B| = 6$, $I = V(G) \setminus B$, and $H = G[I]$. Every vertex of I has degree three in G: a degree-two vertex on a cycle is boundary. The degree-three sector theorem gives independent neighbors, a unique first neighbor towards each other target, and exactly two boundary vertices in each of the three first-step sectors. An edge of H therefore has exactly two boundary coordinates increasing, two decreasing, and two unchanged.

For a fixed $u \in I$, let its sectors be S_v, indexed by its neighbors. Form a quotient graph Q_u on the three neighbors, putting an edge between v, w when there is an edge between S_v, S_w in $G - u$. The quotient is connected. Put

$$P_v^w = \{x \in S_v : d(w, x) = d(u, x)\}.$$

Then P_v^w is nonempty exactly when vw is an edge of Q_u. One direction follows from a horizontal cross-sector edge. For the other, a shortest w–x path of length $r = d(u, x)$ starts at root level one and ends at level r. If n_-, n_0 count its inward and horizontal steps, then $2n_- + n_0 = 1$. Its sole horizontal step must cross directly from S_w to S_v, since vertical edges preserve sectors.

The sector-flux proof gives a boundary vertex in P_v^w and one in $S_v \setminus P_v^w$ whenever P_v^w is nonempty. To recall the sign argument, edges leaving the first set are nonincreasing in distance from u, with at least one strictly decreasing; summing scores cancels internal edges. For the complement use distance from w, with the strict edge vu. Because S_v has exactly two boundary points, precisely one lies in each set.

If a neighbor w of u is interior, the edge uw has two horizontal boundary coordinates. Their number equals $\deg_{Q_u}(w)$, so this degree is two. If a neighbor v is boundary, it belongs to none of the sets P_v^w, since independent root neighbors give $d(w, v) = 2$. Quotient degree two would place its sole sector-mate in both such sets, giving one closer and two equidistant neighbors at u and hence a positive score. Thus boundary neighbors have quotient degree one. The connected quotient is consequently P_3 or K_3, and

$$\deg_H(u) \in \{1, 3\}. \tag{27}$$

In particular, an H leaf has exactly two boundary neighbors, and an internal H vertex has none. Also $|I|$ is even.

Let $F(x) = \sum_{b \in B} d(x, b)$. It is constant on each H component. For an edge from $u \in I$ to $v \in B$, the six changes consist of two -1's, one zero, and three $+1$'s, so $F(v) = F(u) + 1$. At an internal H vertex, the six nonpositive scores sum to zero. At an H leaf, their sum is -2 and each of its two adjacent boundary targets gives score -1. Hence

$$\boxed{\beta_G(u, b) = -\mathbf{1}_{u \sim b} \qquad (u \in I,\ b \in B).} \tag{28}$$

Equivalently, $x \mapsto d(x, b) - \mathbf{1}_{x=b}$ is harmonic at every interior vertex. This exact equality is stronger than the defining nonpositive inequality and is the source of the reduction below.

17.2 Propagation of tight boundary pairs

The boundary profiles satisfy

$$d(x, y) = \max_{b \in B} |d(x, b) - d(y, b)|, \qquad \forall b\ \exists c :\ d(x, b) + d(x, c) = d(b, c).$$

These properties also follow directly by extending a geodesic away from its initial target while its endpoint is nonboundary. A nonboundary endpoint has a farther neighbor, since it has a closer

one and nonpositive score. Strictly increasing distance forces the extension to terminate at a boundary vertex.

There is no five-cycle in H. For each boundary coordinate its horizontal edges form a matching: two horizontal neighbors would leave one closer neighbor and give positive score. On a five-cycle the horizontal count is odd and at most two, hence one. Six coordinates would give six horizontal edge incidences, while the five edges of H each contribute two, giving ten. This is impossible.

At an internal H vertex u, suppose b, c form a tight pair. Its nonnegative defect

$$g_{bc}(x) = d(x, b) + d(x, c) - d(b, c)$$

has value zero at u. Equation (28) implies $\sum_{x\sim u} g_{bc}(x) = 0$, so the pair remains tight at every neighbor. Every coordinate has increments $-1, 0, 1$ at u, and each incident edge is horizontal in exactly two boundary coordinates. A tight partner must have the same horizontal edge. Thus the six coordinates are partitioned into three unique tight pairs. This perfect matching agrees across an edge between two internal H vertices.

Consider such an edge uv, with other neighbors x, y of u and p, q of v. On all these vertices, the paired coordinates are complementary. Select one coordinate from each pair, translate, permute, and reverse coordinate signs. The exhaustive normal form is

$$u = (0, 0, 0), \quad v = (0, 1, 1), \quad x = (1, 0, -1), \quad y = (-1, -1, 0),$$

$$p = (\varepsilon, 1, 2), \quad q = (-\varepsilon, 2, 1), \qquad \varepsilon \in \{-1, 1\}.$$

Both sign cases give $d(y, p) = d(x, q) = 2$ by profile isometry. A common neighbor of y, p cannot be u, v or an interior vertex: the first choices create a triangle through an interior vertex, and the last creates a five-cycle $yuvpw$ in H. It is therefore boundary. By (27), y, p are H leaves. The same holds for x, q. Every internal-to-internal edge therefore exhausts a six-vertex double-star component. Consequently

$$H \text{ has only components } K_2,\ K_{1,3},\ \text{or a six-vertex double-star.}$$

If ℓ is its number of leaves, then $|I| \le 3\ell/2$ and $|I|$ is even. Since $2\ell = |E(I, B)| \le 18$, this already proves the unconditional finite bound $|V(G)| \le 18$.

17.3 A sharper hand reduction to order fourteen

The double-star can in fact be excluded without enumeration. If $\varepsilon = 1$, every selected-coordinate difference between y, p has absolute value two, as do the complementary differences. A common boundary neighbor would have distance one from both and hence give a zero difference, a contradiction.

Suppose $\varepsilon = -1$. Write the actual coordinate values at u as D_1, D_2, D_3 and their complementary values as D'_1, D'_2, D'_3. A boundary neighbor w common to y, p, and a boundary neighbor z common to x, q, must be the two endpoints of pair one, forcing $D_1 = D'_1 = 2$. Positivity on the six interior vertices gives

$$D_2, D_3 \ge 2, \qquad D'_2, D'_3 \ge 3.$$

Each H leaf needs a second boundary neighbor. The only coordinate that can equal one forces respectively

$$x \sim c, \quad y \sim a, \quad p \sim c', \quad q \sim a', \qquad D_2 = D_3 = 2, \quad D'_2 = D'_3 = 3,$$

where (a, a') and (c, c') are pairs two and three. Now $d(x, a) = 2$. Among x's neighbors u, z, c, the first has distance two from a, and $d(z, a) \ge 3$ because $z \sim q$ and $d(q, a) = 4$. Therefore

$c \sim a$. The symmetric argument at p forces $c' \sim a'$. But $d(x, c') = 4$, while the distances to c' from x's three neighbors are $3, 5$, and at most three: the last bound follows from $zqa'c'$. Thus $\beta_G(x, c') \geq 12 - (3 + 5 + 3) = 1$, a contradiction. We have proved that H has only edge and three-leaf-star components.

There is at most one star component and it precludes every other component. Its three leaves have distinct boundary neighbors: a shared boundary point would give two length-two geodesics between interior leaves. Hence its center has boundary profile $(2, 2, 2, 2, 2, 2)$. Two centers would violate injectivity. For a separate edge component uv, its four boundary neighbors are distinct. The two missing boundary coordinates have equal values r, s at u, v. Equation (28) makes the corresponding distances from the two boundary neighbors of u equal $r - 1, r + 1$, and similarly for s. The star center implies that all boundary distances are at most four, so $r, s \leq 3$. Every coordinate of u is now between one and three, putting it at sup-distance at most one from the star center. That would make the two components adjacent, a contradiction.

Thus either $H = K_{1,3}$ and $|V(G)| = 10$, or H is a matching of at most four edges, because each edge component uses four incidences at the six boundary vertices. In all cases

$$\boxed{|V(G)| \leq 14.}$$

An independent standard-library BFS checker exhausts all 60,070 isomorphism classes of two-connected subcubic graphs on orders six through fourteen, generated by nauty. Exactly twelve have boundary six, and they are precisely the cores in Theorem 17.1. The larger census through order eighteen, made in the earlier session, is a separate corroboration. The order bound is proved before either census is used. The census through order fourteen was repeated without nauty for this release (Section 19).

17.4 Exact block assembly and the 201 kernels

At intrinsic boundary at most five, every two-connected subcubic block is all boundary, by the degree-three obstruction. Its order is at most five. The full palette therefore consists of all two-connected subcubic graphs on three through six vertices, and the exceptional cores of orders eight and ten: twenty nonbridge types in total.

Distinct nonbridge blocks of a subcubic graph are vertex-disjoint. Bridges attach only at intrinsic degree-two block vertices, at most one per vertex. At such a vertex the intrinsic stability is one or two; one external bridge lowers it by one. Let h_C count attached sites of intrinsic stability one, and let L count ordinary leaves outside the blocks. The exact assembly condition is

$$6 = L + \sum_C (b(C) - h_C). \tag{29}$$

Choose any block tree from the palette, with ordinary vertices of degrees one or three, attach its bridges injectively to available degree-two sites, and impose (29). Together with arbitrary bridge subdivision, this is a necessary-and-sufficient constructive description.

For the reduced-order bound, write $R = \sum_C (b(C) - 2)$ and let $\Delta_* \geq 0$ be the total cut defect in the exact block budget. Then $R + \Delta_* = 4$. Every palette block satisfies $|V(C)| \leq 3(b(C) - 2)$, with equality only for a triangle. If A counts ordinary degree-three vertices, each contributes at least one cut defect, so $A \leq \Delta_*$. Since $L \leq 6$,

$$|V(K)| = \sum_C |V(C)| + A + L \leq 3R + \Delta_* + 6 = 18 - 2\Delta_* \leq 18.$$

Equality forces four triangles, six ordinary leaves, no ordinary branch vertex, and every triangle site buried by a bridge. The contracted tree has four trivalent internal nodes, whose induced tree is P_4 or $K_{1,3}$. These give exactly the two equality kernels.

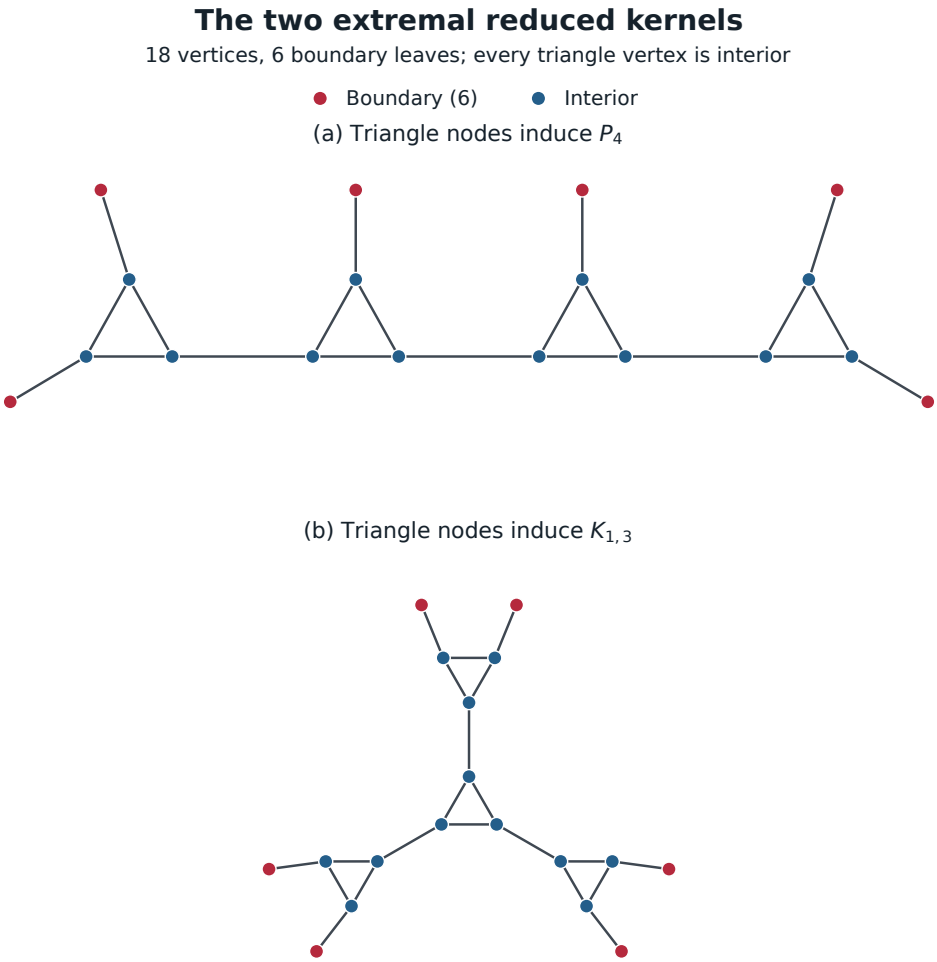


Figure 6: The two eighteen-vertex reduced kernels attaining the sharp subcubic six-boundary bound. Their boundary consists of the six leaves. Arbitrary subdivisions of the displayed bridges preserve the boundary.

The finite closure uses rooted block attachment and insertion of a new trivalent branch on a bridge, retaining maximum degree three and boundary at most six. The usual end-block induction proves completeness. There are 264 states of boundary at most six, of which 201 have boundary six. Their order distribution is

n	6	7	8	9	10	11	12	13	14	15	16	17	18
Number	16	8	10	9	23	20	32	25	27	15	11	3	2

The independent verifier imports no generator and uses no orbit pruning. It checks all 47,332 degree-admissible vertex/root attachments, all 1,257 bridge branchings, fresh boundary scores and reducedness for every state, well-founded generation of every nonseed, and exact canonical closure. The 63 states with boundary at most five agree exactly with the previous subcubic catalogue. The whole subcubic six-boundary proof is not claimed to be formalized in Lean; its sector input is formalized, its new reductions are conventional proofs, and its finite catalogues have exact reproducible checkers.

18 An infinite obstruction to purely local six-boundary tests

The general rooted reduction does not settle global compatibility. The following theorem makes the obstruction explicit, at unbounded diameter. The local model here is exactly the radius-two induced-frame model with minimum degrees, harmonic inequalities, all selected-target descents, and the two-path condition used in the experimental extension.

Theorem 18.1 (All-root local feasibility does not imply realizability)**.** *For each integer $t \geq 3$, let M_t be the metric on $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ with*

$$M_t(a_i, b_j) = t, \qquad M_t(a_i, a_h) = M_t(b_i, b_h) = 2t - 1 \quad (i \neq h).$$

At every designated root, M_t has a feasible radius-two local model satisfying all the conditions just listed, including every nonpositive interior score toward every designated or selected local target. Nevertheless, no finite simple connected graph has exactly these six harmonic boundary vertices with boundary metric M_t.

Proof. First describe its entire positive profile frame. Each part has a unique coordinate at most $t - 1$. Otherwise its coordinates could have no tight partner; two such small coordinates would violate their sum inequality. Write the small coordinates as a at a_i and b at b_j. Every other coordinate in its part must be at least $2t - 1$ minus that minimum. It is therefore at least t, so it cannot have a cross-part tight partner. Its only possible tight partner is its own part's minimum. Thus every positive profile is exactly

$$\begin{array}{l} p(a_i) = a, \quad p(a_h) = 2t - 1 - a \quad (h \neq i), \\ p(b_j) = b, \quad p(b_h) = 2t - 1 - b \quad (h \neq j), \end{array} \qquad 1 \leq a, b \leq t - 1, \quad a + b \geq t.$$

Denote this profile by $p(i, j; a, b)$. Conversely each displayed profile satisfies all frame conditions. At root a_i, select the root and the six profiles

$$p(i, j; 1, t - 1), \quad p(i, j; 2, t - 2) \qquad (j = 1, 2, 3).$$

They induce a K_4 on the root and its three first-step vertices, with one pendant second-step vertex at each first-step vertex. The root degree is three and the three constrained interior degrees are four. There are three disjoint layer-one to layer-two edges. For the first-step vertex indexed by j, its scores toward a_i, any other a_h, b_j, and any other b_h are respectively $0, 0, -2, 0$.

Toward the selected local vertices, the only negative scores are at itself and its own pendant, and all other scores are zero. Every target has a closer selected neighbor. The six-sector rule for degree-three interiors is vacuous. Symmetry gives a valid model at all six roots, uniformly in t.

Suppose a graph realizing this exact boundary exists. Its distance profiles form an isometric induced subgraph of the full frame. A length-t geodesic from a_i to b_j forces every profile $p(i, j; k, t-k)$, $1 \le k \le t-1$: at its kth step both endpoint distances are below t, so they are the unique small coordinates. Hence every diagonal profile $a + b = t$ is present.

Consider such a forced profile with $a = k \le t-2$, and its score toward a_h with $h \ne i$. Its predecessor on the diagonal, or a_i when $k = 1$, contributes -1; its forced successor contributes $+1$. For $k > 1$, the only possible additional positive contributors are profiles with coordinates $(k+1, t-k)$ or $(k+1, t-k+1)$; the profile $(k, t-k+1)$, if present, contributes zero. At $k = 1$, the other first-step vertices also contribute zero, while all the profiles $p(i, j'; 2, t-1)$ contribute $+1$. An interior score cannot be positive. Thus every additional positive contributor is absent. Applying this at all diagonal profiles deletes the entire two layers $a + b = t + 1$ and $a + b = t + 2$.

Every profile edge changes a and b by at most one. If an index changes, the corresponding two small coordinates must both equal $t-1$, which also respects that bound. Designated points meet only diagonal profiles. The two missing layers therefore separate every higher profile from the boundary. Connectivity excludes all higher profiles, leaving precisely the six designated points and the forced diagonals.

This remaining graph consists of six K_4 clusters, with ports P_{ij} at the A side and Q_{ij} at the B side, joined by nine internally disjoint chains of length $L = t-2$. For $i \ne i'$ and $j \ne j'$,

$$\|P_{ij} - Q_{i'j'}\|_\infty = t-1, \qquad d(P_{ij}, Q_{i'j'}) = t.$$

Indeed, using one chain requires a turn in each endpoint cluster, for a total of $L + 2 = t$, and this is attained. Any other route uses at least three chains and costs at least $3L \ge L + 2$. The discrepancy contradicts isometry of the boundary-profile embedding. This proves the impossibility for every $t \ge 3$. □

The positive frame has $9t(t-1)/2$ profiles; the forced diagonal graph has $9t - 3$ vertices. The smallest member was found by a local metric search; its full 33-vertex frame is also independently rejected by an SMT model. The theorem's proof is the uniform argument above, not that solver result. The standard-library checker regenerates the full frame directly for $t = 3, 4, 5$, checks local score coefficients and layer exclusions for $3 \le t \le 12$, and checks the final distance discrepancy by fresh BFS. This identifies a precise limitation of the existing local model, rather than extrapolating a six-boundary classification from small graphs.

Corollary 18.2 (The same local obstruction at every larger boundary size)**.** *For every $k \ge 6$ there are integer metrics on k designated points, of unbounded diameter, which pass all the specified radius-two local tests at every root but cannot be harmonic boundary metrics of any connected graph.*

Indeed, use parts of sizes $p, q \ge 3$ with $p + q = k$, cross distances t and within-part distances $2t - 1$. The same profile, layer, and isometry arguments apply. At an A-root the local model is a K_{q+1} with a pendant second-step vertex at each nonroot vertex. Its first-layer interior degrees are $q + 1 \ge 4$. The score toward the matching opposite target is $1 - q$; the other designated-target scores are zero. The symmetric description holds at a B-root. Thus the local witnesses meet the same constraints, while the two missing layers force the same global distance discrepancy.

19 Verification, dependencies, and further questions

19.1 Exact computational scope

All finite decisions use integer breadth-first distances and integer stability sums. The package directory `checks/` contains the checkers and data used below; `checks/run_checks.py` replays them in a temporary copy. The checks needed by the proofs are: the exact inverse matrices of the three cones; the capped local certificate, with its 835 contradiction trees, 306 feasible selections and a regeneration of all 1,613,472 coefficient choices; the fifteen residual metrics; the complete block palette through order six; the 227-kernel closure; and, for the extensions, the grammar closures, counting identities and audits cited in their sections. The subcubic core census and the 201-kernel closure were carried out with nauty 2.8.9 [7]; their checkers are included and run when the nauty programs are supplied.

The following enumerations were written independently for this release and share no code with the other checkers. Graphs were generated by adding vertices (or, for two-connected graphs, ears) and deduplicated by exact isomorphism tests.

- Every connected graph of order at most eight with five boundary vertices reduces to one of the 119 catalogue kernels of order at most eight, and every such kernel occurs.
- Every connected subcubic graph of order at most eleven with six boundary vertices reduces to one of the 86 catalogue kernels of order at most eleven, and every such kernel occurs.
- At orders $6, \dots, 14$ the numbers of two-connected graphs of maximum degree at most three are 10, 18, 49, 115, 349, 1011, 3421, 11679 and 43418; these 60,070 graphs agree with the nauty census, and exactly twelve of them, the cores of Theorem 17.1, have six boundary vertices.
- Starting from the two-connected blocks with boundary at most six found by that census, and from K_2, the attachment closure of Theorem 8.3 produces exactly 264 subcubic reduced graphs with boundary at most six, of which exactly 201 have boundary six; they are the catalogue. The same closure started from all two-connected graphs of order at most six with boundary at most five produces 264 reduced graphs, of which exactly 227 have boundary five; they are the catalogue.
- For $n \le 13$, the minimum boundary of an n-vertex two-tree and the number of minimizers agree with Theorem 10.1 and with the counts $2, 7, 3, 18, 3, 36, 3, 66$ at $n = 6, \dots, 13$ given by Section 12.

These enumerations check the catalogues and the implementations; the order bounds that make the catalogues complete are the proved theorems.

19.2 Formal verification: precise scope

The Lean project in `lean/` uses Lean 4.35.0 and its standard library; it pins Mathlib for uniformity with the other projects of this repository but imports none of it. The audit module `HarmonicBoundary.AxiomAudit` enumerates every declaration defined in the project modules (15,771 declarations, including generated ones) and checks that each depends only on `propext`, `Classical.choice` and `Quot.sound`. There is no `sorry`, no added axiom and no `native_decide`; finite facts are proved by kernel-checked `decide` or `omega`.

- `DistanceCertificate` defines walks of a given length and proves that a table satisfying local conditions is the shortest-walk metric.
- `RootSectors` proves Theorem 2.2 at graph level: from a finite simple graph, certified distances, a nonboundary vertex of degree three and connectivity after deleting it, it constructs a duplicate-free list of six boundary vertices.
- `WedgeDistance` and `WedgeBoundary` prove the distance and stability formulas for a one-point union of two graphs, the binary case of Theorem 6.1. `BlockBudget` proves the accounting

identity and the kernel bounds from an explicit accounting record; it does not construct the block-cut tree.
- `Sharp14` certifies the distances of Figure 1 and that exactly vertices 9–13 are boundary.
- The 42 modules `LocalSystems/` state and prove the 835 local infeasibility systems of Section 5, each with its Boolean bounds, degree and score inequalities and two-path condition written in the statement. The link between these statements and the profile certificate is made by the certificate checker, not in Lean.
- `RootedMetricTruncation` proves the rooted metric theorem of Section 15, with both sharp constants, on an arbitrary index type; `BipartiteSubcubic` proves the boundary characterization of Section 16 at graph level.
- `EvenEqualityAutomaton` and `OddEqualityGrammar` prove the weight bounds for all runs and derivations of the two-tree grammars, and the degree bound six; `OddCountingIdentities` proves the polynomial identities behind the counting formula. The correspondence between graphs and grammar derivations is a conventional proof.

The metric-cone coverage, the capping reduction, the diameter-three argument, the general block decomposition, the two-tree charging argument, the subcubic interior reduction and the catalogue enumerations are not formalized.

19.3 Rejected shortcuts and remaining extensions

Two tempting reductions are false. A theta graph with three internally disjoint terminal paths of lengths $1, 4, 4$ has boundary six and diameter four, so a proposed general bound $b(G) \geq 2\,\mathrm{diam}(G)$ for two-connected graphs fails. Also, take a pentagon with leaves attached at three vertices, two of them adjacent. The endpoints of the edge joining those two vertices are interior. Contracting that edge produces a square and increases boundary five to boundary six. An explicit adjacency certificate is included. Thus arbitrary interior-edge contraction cannot justify the finite core bound.

The former unbounded-diameter obstruction is closed by Theorem 5.1. The large order and diameter searches are independent audits and discovery tools; they are not needed to justify unbounded completeness. The essential finite dependency is the capped local certificate. The small residual metrics have hand exclusions.

The previous open equality cases in the two-tree bound are now closed by Section 11; Section 12 gives the exact unlabelled count at every order. The complete connected subcubic six-boundary classification is proved in Section 17, including the sharp order bound and both equality kernels. Sections 15 and 16 give all-radius and all-boundary-size extensions. The stated radius-two approach has a proved global compatibility limitation at every boundary size at least six, by Section 18.

The remaining limits are specific. Six-boundary graphs having maximum degree at least four are not classified here; no general order bound for their two-connected cores is proved. The exact block budget reduces that question to the missing core palette. The complete five-boundary graph-to-cone-to-kernel chain, the new subcubic six-boundary classification, and the two-tree graph-to-grammar and unlabelled-counting correspondences are not formalized end to end in Lean. At every fixed two-tree excess, rational unlabelled enumeration is proved by a terminating finite procedure; explicit simplified formulas for $e \geq 2$ are not computed here, and no general quasipolynomial or polynomial-growth assertion is made. The metric truncation theorem preserves the specified neighborhoods about one chosen root; simultaneous all-root compatibility is a distinct question. No external peer review or exhaustive priority claim is made.

20 Research process and AI disclosure

This is fully autonomous agentic research. The results, their proofs, the finite certificates and checkers, the original Lean development and the literature search were produced by autonomous AI research agents in an earlier session; its records do not name the system used. The audits described in those records were carried out within the same automated workflow.

Opus 5.5 by Anthropic used with Claude Code assessed the work for this release. It identified the source problem in the paper of Chiem, Dudarov, Lee, Lee and Liu, read the proofs of the diameter theorem, the capping reduction and the subcubic interior reduction, reran the checkers listed in the previous section, wrote and ran the independent enumerations listed there, arranged the Lean files as one project with a whole-project axiom audit, ran the fresh build, and edited this manuscript. The nauty-based census and closure were not rerun with nauty; the independent census and closure replace them in this release. The remaining sections were carried over and read, but not all were rederived line by line.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed; AI checks are not human peer review, no expert validation is claimed, and the literature comparison is bounded and does not establish priority.

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