Finite real counterexamples and spectral obstructions for q-permanent monotonicity

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Abstract

For an ordered real matrix, write $P_q(A) = \sum_\sigma q^{\mathrm{inv}(\sigma)} \prod_i a_{i,\sigma(i)}$. We give an explicit integer Gram matrix of order 896 and rank three whose derivative at $q = 1$ is strictly negative. Its certificate uses a four-row symmetry and exact integer polynomial arithmetic; all input vectors are listed. An explicit scalar multiple plus the identity is an integer positive-definite counterexample, with a fully specified rational decreasing pair in $(0, 1)$, and identity padding gives real positive-definite counterexamples in every order $N \geq 896$. This answers the real-symmetric case of the Bapat–Lal monotonicity conjecture on $[-1, 1]$ negatively, and refutes da Fonseca's extended conjectures inside the original interval. We also derive the complete limiting endpoint functional for bounded fixed-rank perturbations $I + (t/N)VV^T$. For nonzero limiting covariance, below $t\lambda_{\max}(\Sigma) = 1$ the endpoint derivative is eventually positive for every ordering. Convergence of the ordered empirical measure gives an explicit positive first-order limit. An operator factorization of the complete endpoint score yields a stronger spectral obstruction: a negative limit at scale $N^2 \operatorname{per}(A_N)$ requires at least three covariance eigenvalues strictly larger than $1/t$. If $t\lambda_1(\Sigma) > 1 \geq t\lambda_3(\Sigma)$, the normalized derivative instead has an explicit strictly positive lower bound, for every ordering; this includes every supercritical real perturbation of latent rank two. For a convergent supercritical row cloud, uniformly random orders have a fixed positive normalized derivative except on an exponentially small fraction of labelled permutations. Above the threshold, anisotropic spherical families yield correlation counterexamples with off-diagonal entries $O(N^{-1})$, least eigenvalue tending to one, and condition number tending to 227 in latent rank four or 20001 in latent rank three. A Lean 4 development proves the endpoint identity, the soundness of the integer certificate and the reductions to rational positive-definite counterexamples in every order $N \geq 896$; the finite integer comparison itself is checked by two independent exact programs, and the analytic asymptotic proofs are conventional.

1 Results and context

For $\sigma \in S_n$, let

$$\mathrm{inv}(\sigma) = \#\{(i, j) : i < j,\ \sigma(i) > \sigma(j)\}, \qquad P_q(A) = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

The convention is $0^0 = 1$, and $D_q(A) = \frac{d}{dq} P_q(A)$. The fixed order matters. A simultaneous relabelling by an arbitrary permutation generally changes this polynomial. At $q = -1, 0, 1$ it gives the determinant, the diagonal product, and the permanent, respectively. Bapat and Lal's monotonicity conjecture asks whether a non-diagonal positive-definite matrix has strictly increasing q-permanent on $[-1, 1]$ [2, 3]. Mitchell proved the equivalence with a positive-semidefinite formulation with positive diagonal, and established rank-one and order-three cases [12].

Da Fonseca conjectured strict monotonicity of the q-permanent of a real symmetric positive-definite matrix on an interval with left endpoint at or below -1 and right endpoint $+\infty$ [4, Conjecture 1], and restated it with left endpoint strictly below -1 [5, Conjecture 2]; both contain the real-symmetric case of the Bapat–Lal assertion on $[-1, 1]$.

Kitamura's public Lean proof supplies a complex Hermitian order-144 counterexample by a negative derivative at one [8]. The public problem record, consulted on 1 November 2026, explicitly distinguishes that result from the real-symmetric case, which it lists as unsettled [13]. A companion paper gave real positive-definite counterexamples to da Fonseca's conjectures in the extended range $q > 1$ [7]; it left the real-symmetric question on $[-1, 1]$ open. The present paper concerns that original interval, with a negative derivative at its positive endpoint and strict decrease at two parameters in $(0, 1)$. The real-symmetric case of the Bapat–Lal conjecture is therefore false, and da Fonseca's conjectures fail inside the original interval as well as above it. The mean-field theory below additionally explains an obstruction to arbitrarily weak perturbations within the fixed-rank mechanism.

Theorem 1.1 (A finite real endpoint counterexample)**.** *The ordered integer rows in Appendix A, interpreted by the four-row rule in Section 3, define a Gram matrix $B \in \mathbb{Z}^{896\times 896}$ of rank exactly three, with positive diagonal, such that*

$$D_1(B) < 0.$$

The stronger exact certificate margin is recorded in Table 1. In particular there are a real positive-definite matrix of this same order and rational $0 < q_1 < q_2 < 1$ with $P_{q_2}(A) < P_{q_1}(A)$.

The positive-definite and rational-parameter conclusion is elementary: the endpoint derivative is continuous in the entries and in q; adding a sufficiently small positive diagonal preserves its strict negative sign, and a small interval immediately to the left of one contains a rational decreasing pair. Section 4 strengthens this to a specific integer positive-definite matrix and a specified pair. It follows immediately that real positive-definite counterexamples exist in every order $N \geq 896$: use $A \oplus I_{N-896}$. Every nonzero permutation term fixes the trailing identity coordinates, and these coordinates introduce no inversions, so $P_q(A \oplus I_{N-896}) = P_q(A)$ for every real q.

Theorem 1.2 (Bounded-condition asymptotic families)**.** *There are explicit real algebraic positive-definite correlation matrices C_N in every sufficiently large order such that*

$$\limsup_{N\to\infty} \frac{D_1(C_N)}{N(N-1)\operatorname{per} C_N} < -\frac{1}{40000}, \qquad |(C_N)_{ij}| \leq \frac{904}{N} \quad (i \neq j),$$

and

$$\lambda_{\min}(C_N) \longrightarrow 1, \qquad \kappa_2(C_N) \longrightarrow 227.$$

In particular $\kappa_2(C_N) < 228$ eventually. Before diagonal normalization, the matrices are $I_N + (904/N)V_N V_N^T$ with four real columns in V_N.

A second four-column family has limiting condition number 256 *and the larger normalized derivative margin $-1/1000$.*

With three columns instead, another explicit family has off-diagonal bound $60000/N$, limiting condition number 20001*, and normalized endpoint derivative limit less than $-1/800$.*

These are eventual-order conclusions, with no new explicit sufficient order claimed. They are not assertions of convergence to the identity in operator norm: finitely many eigenvalues remain separated from one. They are also different families from the finite certificate. The full proofs, including the algebraic grids and all constants, are Theorems 10.2, 11.2, and 11.3.

Theorem 1.3 (A strict fixed-rank obstruction)**.** *Let uniformly bounded real rows of a fixed latent dimension satisfy $N^{-1}V_N^TV_N \to \Sigma \neq 0$. For fixed $t > 0$, if $t\lambda_{\max}(\Sigma) < 1$, then*

$$D_1(I_N + (t/N)V_NV_N^T) > 0$$

for all sufficiently large N, for every sequence of row orderings. The same holds after correlation normalization. With convergence of the ordered empirical measure, an explicit strictly positive limit of $D_1/(N\,\mathrm{per})$ is given in Theorem 6.3.

Thus limiting condition number below two is an obstruction for this bounded-row, fixed-rank scaling. This theorem does not extend its conclusion to every positive-definite matrix of condition number below two. Nor is the coherent-saddle transition automatically the transition in the sign of D_1. Indeed, Corollary 6.2 shows that a limiting row law independent of position gives a strictly positive leading endpoint derivative even above that threshold. The complete ordered functional therefore distinguishes these positive configurations from the counterexample orderings.

Theorem 1.4 (A third-eigenvalue obstruction)**.** *Let $r \geq 1$ be fixed, $\|v_{i,N}\| \leq L$, and $N^{-1}V_N^TV_N \to \Sigma$. List the covariance eigenvalues in decreasing order, setting $\lambda_3(\Sigma) = 0$ when $r < 3$. For fixed $t > 0$, if*

$$t\lambda_1(\Sigma) > 1 \qquad \text{and} \qquad t\lambda_3(\Sigma) \leq 1,$$

then, for every sequence of row orderings,

$$\liminf_{N\to\infty} \frac{D_1(I_N + (t/N)V_NV_N^T)}{N(N-1)\,\mathrm{per}(I_N + (t/N)V_NV_N^T)} \geq \frac{(t\lambda_1(\Sigma) - 1)^2}{64r\,t^2L^4} > 0.$$

The same normalized ratio is unchanged by correlation normalization.

Thus a large largest eigenvalue alone is insufficient for the negative endpoint mechanism. For a strictly negative N^2-scale normalized liminf, at least three covariance modes must exceed the threshold $1/t$. The theorem allows equality for the third eigenvalue and places no upper bound on the first two eigenvalues. In latent rank two its explicit lower bound has denominator $128t^2L^4$; together with Theorem 1.3, this gives eventual positivity at every noncritical strength. Section 12 proves the stronger inertia statement behind this obstruction and treats its exact scope.

Contribution and verification scope. The Gaussian representation of a permanent is standard; applications to spatial permutations and related asymptotics appear, for example, in Fyodorov and Muirhead [6]. The new results here are the finite real certificate, its symmetry reduction, the complete ordered endpoint functional, the strict subcritical asymptotic, and the stated bounded-condition counterexamples, together with the full-score operator factorization, quantitative contraction gap, and third-eigenvalue obstruction. Literature searches before and after the derivations found no matching prior results in the primary sources examined. This is a bounded novelty assessment, not a certification of priority. The ordinary permanent-on-top and subset-preserving q-permanent inequalities are different statements; counterexamples to them are not used as counterexamples to monotonicity. A public manuscript of liqsweep claims permanent-on-top for real positive-semidefinite matrices of rank at most two [11]. As discussed in Section 12.3, this has a qualitative endpoint consequence after diagonal addition. We do not claim priority for that rank-two qualitative sign. The proof of Theorem 1.4 is independent of that manuscript and gives an explicit asymptotic margin in arbitrary fixed latent dimension. Corollary 13.1 further shows that negative endpoint orders occupy an exponentially small fraction of labelled permutations of a convergent supercritical row cloud. This is a consequence of the product-limit formula and bounded-difference concentration, not a claim that the counterexample orders have disappeared.

2 The endpoint identity

Let $v_1, \dots, v_n \in \mathbb{R}^r$ be ordered rows, $B_{ij} = v_i \cdot v_j$, and $\ell_i(z) = v_i \cdot z$. For a polynomial $p = \sum_\alpha c_\alpha z^\alpha$, its Fischer norm is

$$\|p\|_{\mathrm{F}}^2 = \sum_\alpha \alpha! |c_\alpha|^2.$$

For an exterior-vector-valued polynomial, sum these norms over an orthonormal exterior basis. Put

$$f = \prod_i \ell_i, \qquad G = \sum_{i<j} (v_i \wedge v_j) \prod_{a \neq i,j} \ell_a.$$

Lemma 2.1 (Gram and ordered wedge identities)**.** *One has*

$$\operatorname{per} B = \|f\|_{\mathrm{F}}^2, \qquad 2D_1(B) = \binom{n}{2} \|f\|_{\mathrm{F}}^2 - \|G\|_{\mathrm{F}}^2.$$

Proof. Expanding the product polynomial, a monomial exponent records the number of rows assigned to each coordinate. Pairing two assignments with the same exponent gives $\alpha!$ bijections of equal colors, exactly the terms in the permanent expansion. This proves the first identity, and the same argument applies to complementary row and column lists.

Expand $\|G\|_{\mathrm{F}}^2$ and use

$$\langle v_i \wedge v_j, v_k \wedge v_l \rangle = b_{ik} b_{jl} - b_{il} b_{jk}.$$

For a fixed permutation monomial $\prod_i b_{i,\sigma(i)}$ and a fixed row pair $i < j$, the contribution is $+1$ if $\sigma(i) < \sigma(j)$ and -1 if $\sigma(i) > \sigma(j)$. Summing over row pairs gives $\binom{n}{2} - 2\operatorname{inv}(\sigma)$. Thus

$$\|G\|_{\mathrm{F}}^2 = \sum_\sigma \left(\binom{n}{2} - 2\operatorname{inv}(\sigma) \right) \prod_i b_{i,\sigma(i)},$$

which is the second identity. This is the ordered-wedge formulation used in [8]; our real-rank-three coefficient and polynomial versions are included in the formal development. □

For nonzero rows $f \neq 0$, so its norm and the permanent are positive. Consequently the dimensionless score

$$R(B) = \frac{\|G\|_{\mathrm{F}}^2}{\binom{n}{2} \operatorname{per} B}$$

exceeds one exactly when $D_1(B) < 0$, and

$$\frac{D_1(B)}{n(n-1) \operatorname{per} B} = \frac{1 - R(B)}{4}. \tag{1}$$

Both sides are unchanged by scaling each row by an arbitrary nonzero real number. This follows since every permutation monomial is multiplied by the same positive product of squared row scales.

3 A four-row integer certificate

3.1 The word and its compressed recurrence

For a type (a, b, c), define the two ordered pairs

$$L = ((-a, b, c), (-a, -b, c)), \qquad R = ((a, b, c), (a, -b, c)).$$

Starting with an empty word, read the type list from inner to outer and replace the current word W by L, W, R. This completely specifies the row order. The resulting word has four rows per type.

Write $X = x^2, Y = y^2, Z = z^2$ and

$$p = a^4X^2 + b^4Y^2 + c^4Z^2 - 2a^2b^2XY - 2a^2c^2XZ - 2b^2c^2YZ. \tag{2}$$

The product polynomial is $f(x, y, z) = F(X, Y, Z)$. Its actual wedge has the parity decomposition

$$\begin{aligned} G_{01} &= yz\, U_{01}(X, Y, Z) + xz\, V_{01}(X, Y, Z), \\ G_{02} &= U_{02}(X, Y, Z), \\ G_{12} &= xy\, U_{12}(X, Y, Z) + V_{12}(X, Y, Z). \end{aligned}$$

Initially $F = 1$ and the other five polynomials are zero. All arithmetic below is over $\mathbb{Z}$. Subscripts denote partial derivatives in the squared variables. One outer type updates the six polynomials by

$$F' = pF, \tag{3}$$

$$U'_{01} = pU_{01} + 8ac\{b^2F + 2b^2XF_X + (a^2X + b^2Y - c^2Z)F_Y\}, \tag{4}$$

$$\begin{aligned} U'_{02} = pU_{02} + 8ac\{&(a^2X - c^2Z)F + X(a^2X - b^2Y - c^2Z)F_X \\ &+ Z(a^2X + b^2Y - c^2Z)F_Z\}, \end{aligned} \tag{5}$$

$$U'_{12} = pU_{12} + 8ac\{-b^2F + (a^2X - b^2Y - c^2Z)F_Y - 2b^2ZF_Z\}, \tag{6}$$

$$V'_{01} = pV_{01} + 8a^2bcF, \tag{7}$$

$$V'_{12} = pV_{12} + 4bc(a^2X - b^2Y + c^2Z)F. \tag{8}$$

Lemma 3.1 (Exact recurrence)**.** *Equations* (3)–(8) *give the ordinary product and actual ordered wedge polynomials of the specified row word.*

Proof. For concatenated words with pairs (f, G) and (h, H), separate wedge pairs within each word and crossing between them. The cross sum is $\nabla f \wedge \nabla h$, so concatenation gives

$$(f, G) \parallel (h, H) = (fh, hG + fH + \nabla f \wedge \nabla h).$$

For the negative and positive pairs their products are $q_- = (cz - ax)^2 - b^2y^2$ and $q_+ = (cz + ax)^2 - b^2y^2$; their wedge vectors are $(2ab, 0, 2bc)$ and $(-2ab, 0, 2bc)$. Applying concatenation twice gives

$$f' = q_-q_+f, \quad G' = pG + f(H + E) + \nabla f \wedge K,$$

where direct differentiation gives

$$\begin{aligned} H &= (8ab^2c\,yz,\ 8ac(a^2X - c^2Z),\ -8ab^2c\,xy), \\ E &= (8a^2bc\,xz,\ 0,\ 4bc(a^2X - b^2Y + c^2Z)), \\ K &= 4ac(z(-a^2X - b^2Y + c^2Z),\ 2b^2xyz,\ x(a^2X - b^2Y - c^2Z)). \end{aligned}$$

Use $f_x = 2xF_X, f_y = 2yF_Y, f_z = 2zF_Z$ and collect the five indicated parity classes. This gives the displayed recurrences and proves the assertion by induction. □

For $e \in \{0, 1\}^3$ and $Q = \sum_\alpha Q_\alpha X^{\alpha_0}Y^{\alpha_1}Z^{\alpha_2}$, put

$$\mathcal{N}_e(Q) = \sum_\alpha Q_\alpha^2(2\alpha_0 + e_0)!(2\alpha_1 + e_1)!(2\alpha_2 + e_2)!.$$

Distinct physical exponent parities are Fischer-orthogonal. Hence

$$P := \operatorname{per} B = \mathcal{N}_{000}(F), \tag{9}$$

$$T_U := \mathcal{N}_{011}(U_{01}) + \mathcal{N}_{000}(U_{02}) + \mathcal{N}_{110}(U_{12}), \tag{10}$$

$$T := \|G\|_{\mathrm{F}}^2 = T_U + \mathcal{N}_{101}(V_{01}) + \mathcal{N}_{000}(V_{12}) \geq T_U. \tag{11}$$

The exact integer inequality $T_U > \binom{n}{2}P$ therefore suffices for a negative endpoint derivative. The two correction terms need not be evaluated for that conclusion; they are retained in the independent full-wedge arithmetic check.

3.2 The finite data and sign

Table 1: Exact finite witness. Ratios displayed as decimals are explanatory; the strict rational inequality is checked on the full integers.

Quantity	Value or exact bound
Order / number of four-row types	896 / 224
Rank of the integer Gram matrix	3
Largest absolute row coordinate	32
Largest Gram diagonal entry	2490
Nonzero row minor (one-based indices)	$(417, 418, 477)$, determinant 1012
Number of decimal digits of P	4152
Three-component ratio $T_U/(\binom{896}{2}P)$	$1.0019230541058451\ldots$
Full-wedge ratio $T/(\binom{896}{2}P)$	$1.0019250593839399\ldots$
Exact strict certificate	$1000T_U > 1001\binom{896}{2}P$
Consequence for $D_1(B)/\operatorname{per} B$	$< -5012/25$
Consequence for $D_1(B)/(896 \cdot 895 \operatorname{per} B)$	$< -1/4000$
Explicit positive-definite scaling $M = 10^4 n^3$	7193231360000
Entry bound for $A = MB + I$: $H = 2490M + 1$	17911146086400001

The exact report stores P, T_U, T and all five component norms as decimal integers. The SHA-256 of the inner-to-outer input file is

`7220026a9fb5b2d9e62ef46b9b7196b2ff768b4841f71ccb45275ee445987751`.

The data in Appendix A specify every row, without any optimization routine or floating-point number in their definition. Applying the integer recurrences and factorial sums gives the exact strict inequalities in Table 1. Lemma 2.1 then proves the negative sign. Positive semidefiniteness follows from $B = VV^T$. A nonzero three-by-three minor specified in the table proves rank three; each row is nonzero, so every diagonal entry is positive. This proves Theorem 1.1.

The accompanying package also records the full integer numerators and denominator of the rational pair specified in Section 4. The common denominator has 16834 decimal digits; its compact exact formula is preferable in the mathematical text.

The exact engine compares integers, not the displayed decimal ratios. A second, independent program uses neither the four-row symmetry nor the recurrences above. It expands the ordinary product and ordered-wedge recurrences

$$f' = f\ell, \qquad S'_a = S_a\ell + v_a f, \qquad G'_{ab} = G_{ab}\ell + v_b S_a - v_a S_b$$

row by row in the actual order, where $\ell = v \cdot z$ is the new row form and $S_a = \partial f/\partial z_a$, and evaluates P and the Fischer norms of every exponent-parity class of the three components of G modulo 500 primes between 2^{30} and 2^{31}. Every permutation monomial has absolute value at most $\prod_i b_{ii}$, so $P \le n! \prod_i b_{ii}$ and each parity-class norm is at most $\binom{n}{2} n! \prod_i b_{ii}$. The product of the primes exceeds this bound, which has 15471 bits. The Chinese remainder theorem therefore determines P, T_U and T exactly from the residues, and all three agree with the integer report. The only nonzero parity classes are the five in (9)–(11). This determination is independent of Lemma 3.1; it relies only on Lemma 2.1 and exact arithmetic. The precise Lean proof boundary is given in Section 14.

4 An explicit integer positive-definite counterexample from a finite certificate

This section converts a finite positive-semidefinite endpoint certificate with a specified margin into an integer positive-definite matrix and two specified rational parameters in $(0, 1)$. The estimates below are conventional proofs; their explicit numerical regularization is separate from the scope of the Lean certificates reported elsewhere.

For an ordered k-by-k positive-semidefinite matrix C, write $D_1(C) = P_1'(C)$, $c_k = \binom{k}{2}$, and

$$T(C) = \left\| \sum_{i<j} (v_i \wedge v_j) \prod_{a \neq i,j} (v_a \cdot z) \right\|_{\mathrm{F}}^2, \qquad C = (v_i \cdot v_j)_{i,j}.$$

Thus $2D_1(C) = c_k \operatorname{per} C - T(C)$. The empty permanent is one. For $I \subseteq [n]$, let $D_{\mathrm{emb}}(B[I])$ denote the derivative obtained by allowing permutations on I and fixing every index in $[n] \setminus I$, with inversions counted in the original order on $[n]$.

Lemma 4.1 (A uniform bound for embedded principal derivatives)**.** *Let B be a real positive-semidefinite matrix of order $n \ge 2$, let $I \subseteq [n]$, and put $k = |I|$, $j = n - k$. Then*

$$|D_{\mathrm{emb}}(B[I])| \le \left\{ \frac{c_k + c_k^2}{2} + \frac{jk^2}{2} \right\} \operatorname{per} B[I] \le n^4 \operatorname{per} B[I]. \tag{12}$$

Proof. For $C = B[I]$, define the permanental cofactor matrix

$$F_{ab}(C) = c_{ab} \operatorname{per} C(a \mid b),$$

where $C(a \mid b)$ deletes row a and column b. The matrix $(\operatorname{per} C(a \mid b))_{a,b}$ is a Gram matrix of the products of all but one row form, in the Fischer inner product. The Schur product theorem therefore shows that $F(C)$ is positive semidefinite. The singleton case of Lieb's permanent inequality [10] gives $F_{aa}(C) \le \operatorname{per} C$, and hence

$$|F_{ab}(C)| \le \operatorname{per} C. \tag{13}$$

For each fixed index $u \notin I$, write

$$\operatorname{cut}_u F(C) = \sum_{\substack{a,b \in I \\ a<u<b}} F_{ab}(C),$$

with the labels on I inherited from $[n]$. Inserting the fixed index u contributes twice the number of active indices mapped from its left to its right. Summing over permutations gives the exact identity

$$D_{\mathrm{emb}}(C) = D_1(C) + 2 \sum_{u \notin I} \operatorname{cut}_u F(C). \tag{14}$$

There are at most $k^2/4$ pairs crossing each cut. Thus (13) bounds the second term in (14) in absolute value by $jk^2 \operatorname{per} C/2$.

Each of the c_k summands in the exterior polynomial defining $T(C)$ has squared Fischer norm

$$(c_{aa}c_{bb} - c_{ab}^2) \operatorname{per} C[\{a,b\}^c] \le c_{aa}c_{bb} \operatorname{per} C[\{a,b\}^c] \le \operatorname{per} C.$$

The final inequality follows by applying the singleton form of Lieb's inequality twice. The triangle inequality, or Cauchy–Schwarz, now gives $T(C) \le c_k^2 \operatorname{per} C$. Therefore

$$|D_1(C)| \le \frac{c_k + c_k^2}{2} \operatorname{per} C.$$

The same statement holds for $k = 0, 1$, when both sides involving the derivative vanish. These estimates prove the first inequality in (12). For $k \ge 1$, $(c_k + c_k^2)/2 \le k^4/2$, while $jk^2/2 \le n^3/2 \le n^4/2$. This proves the second inequality; the case $k = 0$ is immediate. □

Theorem 4.2 (Explicit integer regularization)**.** *Let $n \ge 6$, and let B be an integer real positive-semidefinite matrix of order n whose diagonal entries are at least one. Suppose that*

$$T(B) > \frac{1001}{1000} \binom{n}{2} \operatorname{per} B. \tag{15}$$

Set

$$M = 10^4 n^3, \qquad A = MB + I_n, \qquad H_0 = \max_i b_{ii}, \qquad H = MH_0 + 1. \tag{16}$$

Then A is an integer positive-definite matrix, its entries have absolute value at most H, and

$$D_1(A) < -M^n \frac{n(n-5)}{20000} \operatorname{per} B < 0. \tag{17}$$

Its least eigenvalue is at least one, and is exactly one when B is singular. If B has rank $r < n$, then precisely $n - r$ eigenvalues of A equal one.

Proof. Put $\eta = M^{-1}$. Expansion in the diagonal additions gives

$$D_1(B + \eta I_n) = D_1(B) + \sum_{\varnothing \ne J \subseteq [n]} \eta^{|J|} D_{\mathrm{emb}}(B[J^c]). \tag{18}$$

The diagonal assumptions and the iterated singleton form of Lieb's inequality imply $\operatorname{per} B[J^c] \le \operatorname{per} B$ for every J. They also imply $\operatorname{per} B \ge \prod_i b_{ii} \ge 1$. By Lemma 4.1, the sum in (18) has absolute value at most

$$n^4 \operatorname{per} B\big((1+\eta)^n - 1\big).$$

Since $n\eta \le 1/2$, the inequalities $\binom{n}{j} \le n^j$ and the geometric-series formula give

$$(1+\eta)^n - 1 \le \sum_{j=1}^{\infty} (n\eta)^j = \frac{n\eta}{1 - n\eta} \le 2n\eta.$$

The perturbation in the derivative is consequently at most $2n^5\eta \operatorname{per} B = n^2 \operatorname{per} B/5000$. On the other hand, (15) and the endpoint identity imply

$$D_1(B) < -\frac{n(n-1)}{4000} \operatorname{per} B.$$

Combining the last two estimates yields

$$D_1(B+\eta I_n) < -\left(\frac{n(n-1)}{4000} - \frac{n^2}{5000}\right)\operatorname{per} B = -\frac{n(n-5)}{20000}\operatorname{per} B < 0.$$

The derivative is homogeneous of degree n in the matrix entries, so multiplication by M^n gives (17). The remaining assertions follow from $A = MB + I_n$, from the spectral theorem, and from $|b_{ij}| \le \sqrt{b_{ii}b_{jj}} \le H_0$. □

A certificate may establish (15) by computing any explicit lower bound $U \le T(B)$ and checking the integer inequality $1000U > 1001\binom{n}{2}\operatorname{per} B$. In particular, a sum of selected mutually orthogonal parity-component norms is sufficient. The values of n and H_0 are read from the particular frozen row file; none of the preceding estimates depends on a choice between witness versions.

Corollary 4.3 (Specified rational points of strict decrease)**.** *Under the hypotheses of Theorem 4.2, define the positive integer*

$$Q = 4n!\, n^4 H^n, \qquad q_1 = 1 - \frac{2}{Q}, \qquad q_2 = 1 - \frac{1}{Q}. \tag{19}$$

Then $0 < q_1 < q_2 < 1$*, and*

$$P_q'(A) \le -\frac{1}{2} \quad (q_1 \le q \le 1), \qquad P_{q_1}(A) > P_{q_2}(A). \tag{20}$$

Thus the integer matrix A and the two rational parameters in (19) *form a fully specified positive-definite counterexample.*

Proof. The polynomial $P_q(A)$ has integer coefficients. Since $D_1(A) < 0$, it follows that $D_1(A) \le -1$. For $0 \le q \le 1$, termwise differentiation of the permutation expansion and $\operatorname{inv}(\sigma) \le \binom{n}{2} \le n^2$ give

$$|P_q''(A)| \le n!\, n^4 H^n = \frac{Q}{4}.$$

Terms with fewer than two inversions contribute zero, so this bound also holds at $q = 0$. For $q \in [1 - 2/Q, 1]$, the fundamental theorem of calculus gives

$$P_q'(A) \le P_1'(A) + \frac{Q}{4}(1-q) \le -1 + \frac{1}{2} = -\frac{1}{2}.$$

Since $Q > 2$, the indicated interval lies in $(0, 1]$, and integration between q_1 and q_2 proves the strict inequality. □

5 A one-parameter reduction for coherent ordering

The four-row symmetry also permits an exact ordering theorem for a single coherent evaluation. This theorem concerns the actual exterior polynomial, including its correction terms.

Take m triples (a_j, b_j, c_j) of nonnegative real numbers, with $a_j^2 + b_j^2 > 0$ for each triple, and assume that at least one triple has $a_j b_j c_j > 0$. An order of the triples defines a word of $4m$ rows by starting with the empty word and successively prepending $(-a_j, b_j, c_j), (-a_j, -b_j, c_j)$ and appending $(a_j, b_j, c_j), (a_j, -b_j, c_j)$. Indices $j = 0, \dots, m-1$ refer to this inner-to-outer order. For its rows v_i, set

$$f(z) = \prod_i (v_i \cdot z), \qquad G_{ab}(z) = \sum_{i<j} (v_{ia}v_{jb} - v_{ib}v_{ja}) \prod_{h \ne i,j} (v_h \cdot z), \quad 0 \le a < b \le 2.$$

Fix $s > 0$, put $z_s = (1, is, 0)$, and define

$$S_j = a_j^2 + s^2 b_j^2, \qquad t_j = \frac{a_j c_j}{S_j}, \qquad d_j = \frac{a_j^2 - s^2 b_j^2}{S_j}, \qquad K_s = \sum_j \frac{b_j c_j}{S_j},$$

$$A = \sum_j (2j+1) t_j, \qquad B = \sum_j t_j \left(d_j + 2 \sum_{\ell < j} d_\ell \right).$$

The quantities A, B depend on the order; K_s does not.

Proposition 5.1 (Exact coherent evaluation)**.** *One has $f(z_s) = \prod_j S_j^2 > 0$, and*

$$\frac{G_{01}(z_s)}{f(z_s)} = 0, \qquad \frac{G_{02}(z_s)}{f(z_s)} = 4(A+B), \qquad \frac{G_{12}(z_s)}{f(z_s)} = 4K_s + \frac{4i}{s}(B - A). \tag{21}$$

Consequently

$$\mathcal{E}_s := \frac{\sum_{a<b} |G_{ab}(z_s)|^2}{|f(z_s)|^2} = 16 \left((A+B)^2 + \frac{(B-A)^2}{s^2} + K_s^2 \right). \tag{22}$$

At the unit point $z_s/\sqrt{1+s^2}$, the corresponding ratio is $(1+s^2)^2 \mathcal{E}_s$.

Proof. The polynomial identities in the four-row recurrence hold for real inputs as well as integers. Evaluate the squared variables at $(X, Y, Z) = (1, -s^2, 0)$. For the orbit factor p_j,

$$p_j = S_j^2, \qquad \frac{(p_j)_X - s^2 (p_j)_Y}{p_j} = 2, \qquad \frac{(p_j)_X + s^2 (p_j)_Y}{p_j} = 2 d_j.$$

Thus, for the product F of the first j orbit factors,

$$\frac{F_X - s^2 F_Y}{F} = 2j, \qquad \frac{F_X + s^2 F_Y}{F} = 2 \sum_{\ell < j} d_\ell.$$

Substituting in (5)–(6) gives the increments of the normalized ratios

$$\Delta \frac{U_{02} - s^2 U_{12}}{F} = 8(2j+1) t_j,$$
$$\Delta \frac{U_{02} + s^2 U_{12}}{F} = 8 t_j \left(d_j + 2 \sum_{\ell < j} d_\ell \right).$$

Equation (8) gives $\Delta(V_{12}/F) = 4 b_j c_j / S_j$. Summing from the zero initial exterior polynomials and using $G_{02} = U_{02}$, $G_{12} = xy U_{12} + V_{12}$, and $G_{01} = yz U_{01} + xz V_{01}$ proves (21). Its real and imaginary parts give (22). Finally G has degree two less than f, which proves the unit-point scaling. □

Theorem 5.2 (One-parameter sorting)**.** *For every fixed $s > 0$, an ordering maximizing $\mathcal{E}_s$ is obtained by sorting the triples in nondecreasing order of*

$$\frac{a_j c_j}{a_j^2 + \kappa b_j^2} \tag{23}$$

for some $\kappa > 0$, with suitable ordering among ties. The family of sorts is independent of s; its maximizing parameter may depend on s.

Proof. Since $t_j \geq 0$ and $|d_j| \leq 1$, one has $|B| \leq A$. The triple with all entries positive has $t_j > 0$ and $|d_j| < 1$, so its contribution makes $|B| < A$ in every order. The order-independent term K_s^2 can be dropped when maximizing. Write

$$Q(A,B) = (A+B)^2 + s^{-2}(B-A)^2 = D(A^2+B^2) + 2CAB, \qquad D = 1+s^{-2}, \quad C = 1-s^{-2}.$$

Here $D > |C|$, so Q is positive definite.

Choose an order maximizing Q, and denote its pair by (A, B). It also maximizes the linear functional $A' + \lambda B'$, where

$$\lambda = \frac{CA+DB}{DA+CB} \in (-1,1). \tag{24}$$

Indeed, $DA + CB > 0$, and a strict increase in this functional would strictly increase Q by its quadratic expansion. Moreover

$$(DA+CB) \pm (CA+DB) = (D \pm C)(A \pm B) > 0,$$

which proves the strict parameter bounds in (24).

If adjacent types i, j occur in that order, swapping them changes $A + \lambda B$ by

$$2\big((1+\lambda d_j)t_i - (1+\lambda d_i)t_j\big).$$

Both factors $1 + \lambda d_j$ are positive. A strict inversion in the keys $t_j/(1+\lambda d_j)$ would therefore improve the functional, contradicting maximality. The order is sorted in those keys, and

$$\frac{t_j}{1+\lambda d_j} = \frac{1}{1+\lambda}\,\frac{a_j c_j}{a_j^2 + s^2 \frac{1-\lambda}{1+\lambda} b_j^2}.$$

Taking $\kappa = s^2(1-\lambda)/(1+\lambda) > 0$ proves the theorem. □

Corollary 5.3 (A finite sweep, including ties)**.** *At most $\binom{m}{2} + 1$ interval sorts suffice to attain the global maximum in Theorem 5.2. If the row coordinates and s^2 are rational, the sweep and the score comparisons can be carried out with exact rational arithmetic.*

Proof. Put $u_j = a_j c_j$. Two keys that are not identical as functions of κ cross at most once for $\kappa > 0$, at

$$\kappa = \frac{u_j a_i^2 - u_i a_j^2}{u_i b_j^2 - u_j b_i^2},$$

provided the denominator is nonzero and the quotient is positive. These crossings partition $(0, \infty)$ into at most $\binom{m}{2} + 1$ open intervals. Sort once at any parameter in each interval.

To check that crossing ties cannot hide a larger value, let $X = (A, B)$ be a global maximizer and use its gradient functional from (24). Any other ordering Y tied for that functional must have $Y = X$: otherwise the quadratic expansion gives $Q(Y) = Q(X) + Q(Y - X) > Q(X)$. There are finitely many orders. A sufficiently small one-sided perturbation of the parameter cannot promote an order whose supporting value was strictly smaller, so it selects a tied maximizer with the same (A, B). This covers simultaneous crossings as well. Keys identical for all parameters have zero adjacent-swap increment in both A and B, so arbitrary tie-breaking within those classes does not change the score. All crossing parameters, keys, and values of Q are rational under the stated assumptions. □

Individual zero values of a_j, b_j, or c_j are permitted; the assumption of one strictly positive triple ensures $A > |B|$ and keeps the supporting parameter interior. If all $t_j = 0$, then $A = B = 0$ and $\mathcal{E}_s = 16K_s^2$ is order-independent; this trivial case does not need a supporting parameter. Boundary-only systems with $A > 0$ but no strictly positive triple are outside the sorting theorem as stated.

The optimization proved here is at one specified complex point. A larger value of $\mathcal{E}_s$ does not by itself imply a larger coefficient-square Fischer norm or a negative endpoint derivative. The finite signs elsewhere in this paper are certified separately by their exact integer Fischer-norm computations.

6 Finite-rank mean-field perturbations

The next results give an asymptotic criterion for a negative endpoint derivative and a strict obstruction in the subcritical regime. They are conventional mathematical proofs. Their measure and limiting arguments are not asserted to form a completed Lean theorem.

Fix a finite integer $r \geq 1$, a real number $t > 0$, and ordered vectors $v_{i,N} \in \mathbb{R}^r$, $1 \leq i \leq N$, with $\|v_{i,N}\| \leq L$ for a constant $L < \infty$ independent of N. Write V_N for the matrix with these rows and set

$$B_N = V_N V_N^T, \qquad A_N = I_N + \frac{t}{N} B_N, \qquad \Sigma_N = \frac{1}{N} \sum_{i=1}^{N} v_{i,N} v_{i,N}^T. \tag{25}$$

Every A_N is real positive definite. Suppose initially that the ordered empirical measures satisfy

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(i/N, v_{i,N})} \implies \mu \tag{26}$$

weakly on $[0,1] \times \overline{B}(0,L)$. The first marginal of μ is Lebesgue measure, and

$$\Sigma = \int v v^T \, d\mu(x,v) = \lim_N \Sigma_N.$$

The order coordinate x is retained throughout.

For an ordered matrix A, let $D_1(A) = P_1'(A)$, with the inversion-weighted q-permanent as defined in the main text. For Gram rows w_i, put

$$f(z) = \prod_i (w_i \cdot z), \qquad G(z) = \sum_{i<j} (w_i \wedge w_j) \prod_{a \neq i,j} (w_a \cdot z), \qquad T(A) = \|G\|_{\mathrm{F}}^2.$$

Here the Fischer norm is $\| \sum_\alpha c_\alpha z^\alpha \|_{\mathrm{F}}^2 = \sum_\alpha \alpha! |c_\alpha|^2$, summed over exterior components. The identities

$$\operatorname{per} A = \|f\|_{\mathrm{F}}^2, \qquad 2D_1(A) = \binom{N}{2} \operatorname{per} A - T(A) \tag{27}$$

follow by expanding the Fischer inner products: the coefficient of a permutation monomial in $T(A)$ is $\binom{N}{2} - 2\operatorname{inv}(\sigma)$. The ordered wedge identity is also used in Kitamura's public proof package [8]. Consequently, for positive permanent,

$$\frac{D_1(A)}{N(N-1)\operatorname{per} A} = \frac{1}{4} \left(1 - \frac{T(A)}{\binom{N}{2} \operatorname{per} A} \right). \tag{28}$$

6.1 The general endpoint criterion

For $h \in \mathbb{C}^r$, define

$$\Phi(h) = \int \log(1 + t|v \cdot h|^2)\, d\mu(x,v) - \|h\|^2, \qquad K = \arg\max_{h \in \mathbb{C}^r} \Phi(h), \tag{29}$$

$$\ell_h(v) = v \cdot h, \qquad p_h(v) = \frac{t|\ell_h(v)|^2}{1 + t|\ell_h(v)|^2}, \tag{30}$$

$$q_h(v) = \frac{t\overline{\ell_h(v)}\, v}{1 + t|\ell_h(v)|^2}, \qquad m_h = \int p_h(v)\, d\mu(x,v), \tag{31}$$

$$Q_h(s) = \int_{\{x<s\}} q_h(v)\, d\mu(x,v), \qquad Q_h = \int q_h(v)\, d\mu(x,v). \tag{32}$$

The vector-valued function q_h is continuous, including at $\ell_h(v) = 0$. Put

$$W_h = \iint_{\{x<y\}} q_h(v) \wedge q_h(w)\, d\mu(x,v)\, d\mu(y,w) \tag{33}$$

and

$$\mathcal{R}(h) = 2\|W_h\|^2 + 2\int (1 - p_h(v))\|Q_h - 2Q_h(x)\|^2\, d\mu(x,v) + (1 - m_h)^2. \tag{34}$$

Theorem 6.1 (Full mean-field endpoint criterion)**.** *Under* (26)*, K is nonempty and compact, and*

$$\begin{aligned} \min_{h \in K} \mathcal{R}(h) &\le \liminf_{N\to\infty} \frac{T(A_N)}{\binom{N}{2} \operatorname{per} A_N} \\ &\le \limsup_{N\to\infty} \frac{T(A_N)}{\binom{N}{2} \operatorname{per} A_N} \le \max_{h \in K} \mathcal{R}(h). \end{aligned} \tag{35}$$

If $\min_K \mathcal{R} > 1$, then $D_1(A_N) < 0$ for all sufficiently large N. If $\mathcal{R} = \mathcal{R}_*$ on K, then

$$\frac{D_1(A_N)}{N(N-1) \operatorname{per} A_N} \longrightarrow \frac{1 - \mathcal{R}_*}{4}. \tag{36}$$

At every $h \in K$,

$$Q_h = \overline{h}, \qquad m_h = \|h\|^2 < 1. \tag{37}$$

Both Φ and $\mathcal{R}$ are invariant under a global phase of h, and under conjugation because the rows are real. Thus two conjugate phase circles of maxima give a single limiting score. A simpler sufficient criterion is $\min_{h \in K} 2\|W_h\|^2 > 1$. The other two terms in (34) are nonnegative.

Corollary 6.2 (Product ordered limits)**.** *Suppose that the ordered empirical limit is $\mu(dx, dv) = dx\, \nu(dv)$, so that its row law does not depend on the position coordinate. Then, for every $h \in K$,*

$$\mathcal{R}(h) = 1 - \frac{m_h(4 - m_h)}{3}. \tag{38}$$

If $t\lambda_{\max}(\Sigma) > 1$, then

$$\liminf_{N\to\infty} \frac{D_1(A_N)}{N(N-1) \operatorname{per} A_N} \ge \min_{h \in K} \frac{m_h(4 - m_h)}{12} > 0. \tag{39}$$

In particular the endpoint derivative is eventually positive.

Proof. The product form of μ gives $Q_h(s) = sQ_h$ and $W_h = \frac{1}{2}Q_h \wedge Q_h = 0$. At a maximizer, (37) gives $\|Q_h\|^2 = m_h$. Since p_h also depends only on the row, the middle term of (34) is

$$2(1-m_h)\|Q_h\|^2 \int_0^1 (1-2s)^2\, ds = \frac{2}{3} m_h(1-m_h).$$

Adding $(1-m_h)^2$ proves (38).

Let u be a real unit top eigenvector of Σ. The elementary bound $\log(1+s) \geq s - s^2/2$ for $s \geq 0$ and the row bound $\|v\| \leq L$ yield

$$\Phi(\epsilon u) \geq \epsilon^2\big(t\lambda_{\max}(\Sigma) - 1\big) - \frac{t^2 L^4}{2}\epsilon^4 > 0$$

for sufficiently small $\epsilon > 0$. Since $\Phi(0) = 0$, zero does not belong to K. Compactness of K and $m_h = \|h\|^2 < 1$ therefore imply $\min_K m_h(4-m_h) > 0$. Combining (38), (35), and (28) proves (39). □

The hypothesis is a product form of the ordered empirical limit itself; no probabilistic assertion about finite random row orders is used.

6.2 A strict obstruction below the covariance threshold

Theorem 6.3 (Subcritical endpoint increase)**.** *Suppose $\Sigma \neq 0$ and $t\lambda_{\max}(\Sigma) < 1$. Define*

$$R_0 = (I_r - t\Sigma)^{-1}, \qquad M_s = \int_{\{x<s\}} vv^T\, d\mu(x,v).$$

Then

$$\operatorname{per} A_N \longrightarrow \det(I_r - t\Sigma)^{-1} \tag{40}$$

and

$$\frac{D_1(A_N)}{N \operatorname{per} A_N} \longrightarrow 2t^2 \int_0^1 \operatorname{tr}\big(M_s R_0(\Sigma - M_s)\big)\, ds > 0. \tag{41}$$

In particular $D_1(A_N) > 0$ for all sufficiently large N.

Corollary 6.4 (Arbitrary changing orders)**.** *The ordered weak-convergence assumption can be dropped from the strict sign conclusion. If only $\Sigma_N \to \Sigma \neq 0$ is assumed, with the same fixed rank and row bound, and $t\lambda_{\max}(\Sigma) < 1$, then*

$$\liminf_N \frac{D_1(A_N)}{N \operatorname{per} A_N} \geq \frac{t^2(\operatorname{tr}\Sigma^2)^2}{8L^4} > 0. \tag{42}$$

More generally, eventual bounds $t\lambda_{\max}(\Sigma_N) \leq 1 - \varepsilon$ and $\operatorname{tr}(\Sigma_N^2) \geq s > 0$, for fixed $\varepsilon, s > 0$, imply the lower bound $t^2 s^2/(8L^4)$.

For $N > r$, the least eigenvalue of A_N is one, while its largest eigenvalue is $1 + t\lambda_{\max}(\Sigma_N)$. Thus Theorem 6.3 gives a strict obstruction for this entire bounded-row, fixed-rank mean-field construction below limiting condition number two. It is not a theorem for every positive-definite matrix of condition number below two, and does not include the critical boundary $t\lambda_{\max}(\Sigma) = 1$.

7 Proof of the full endpoint criterion

7.1 Gaussian representation and concentration

Gaussian representations of permanents also appear in the analysis of spatial random permutations [6]. We give the specific identities and limiting arguments needed here. Let Z be a standard complex Gaussian vector in $\mathbb{C}^r$, with density $\pi^{-r}e^{-\|z\|^2}$. Expanding by principal subsets and using the Fischer identity gives

$$\operatorname{per} A_N = \mathbb{E}\prod_{i=1}^{N}\left(1+\frac{t}{N}|v_i\cdot Z|^2\right). \tag{43}$$

After $z=\sqrt{N}\,h$, the corresponding probability measure is

$$\Pi_N(dh)=\frac{e^{N\Phi_N(h)}\,dh}{\int e^{N\Phi_N(u)}\,du}, \qquad \Phi_N(h)=\frac{1}{N}\sum_i \log(1+t|v_i\cdot h|^2)-\|h\|^2. \tag{44}$$

The functions Φ_N converge to Φ locally uniformly. Indeed, weak convergence and uniform continuity on compact products give uniform convergence on a finite h-net, and then on each compact set. Uniformly in N,

$$\Phi_N(h)\le \log(1+tL^2\|h\|^2)-\|h\|^2. \tag{45}$$

The right side is coercive and is at most $-\|h\|^2/2$ outside a fixed ball. This proves the compactness and nonemptiness of K.

The measures Π_N concentrate in every neighborhood of K. To see this, take a large ball containing K and the region where the upper bound in (45) may be nonnegative. On the compact complement of a neighborhood of K, Φ is separated from its maximum by a strictly positive amount. Local uniform convergence preserves a smaller gap for Φ_N. A small ball around one maximizer gives a denominator bounded below by its fixed volume times $e^{N(\max\Phi-\epsilon)}$. The outer tail is controlled uniformly by $e^{-N\|h\|^2/2}$. Choosing ϵ smaller than the compact gap proves the assertion. No nondegeneracy of the Hessian is needed.

Differentiating Φ in the real and imaginary coordinates is legitimate by the uniform row bound. Its stationary equation is $Q_h=\overline{h}$. Taking its bilinear scalar product with h gives $m_h=\|h\|^2$. Since $p_h(v)<1$ everywhere, $m_h<1$. This proves (37).

7.2 An exact three-sector identity

Suppress the index N. For a retained set $I\subset[N]$, put $J=I^c$, $k=|I|$, and

$$\begin{aligned} f_I(z)&=\prod_{i\in I}(v_i\cdot z),\\ G_I(z)&=\sum_{\substack{i<j\\ i,j\in I}}(v_i\wedge v_j)\prod_{b\in I\setminus\{i,j\}}(v_b\cdot z),\\ H_{a,I}(z)&=\sum_{i\in I}\operatorname{sign}(a-i)\,v_i\prod_{b\in I\setminus\{i\}}(v_b\cdot z), \qquad a\in J. \end{aligned}$$

Empty products equal one and empty sums equal zero.

Lemma 7.1 (All exterior sectors)**.** *One has the exact identity*

$$T(A_N)=\sum_{I\subset[N]}\left(\frac{t}{N}\right)^{|I|}\left(\|G_I\|_{\mathrm{F}}^2+\sum_{a\in I^c}\|H_{a,I}\|_{\mathrm{F}}^2+\binom{|I^c|}{2}\|f_I\|_{\mathrm{F}}^2\right). \tag{46}$$

Proof. Use Gram rows $w_i = \sqrt{t/N}\, v_i \oplus e_i$. Split their wedge polynomial into the exterior sectors $\Lambda^2\mathbb{C}^r$, $\mathbb{C}^r \wedge \mathbb{C}^N$, and $\Lambda^2\mathbb{C}^N$. In the first sector, the coefficient of the added-variable monomial $\prod_{a\in J} x_a$ is $(t/N)^{k/2} G_I$. In the second, with exterior index $a \in J$, the coefficient of $\prod_{b\in J\setminus\{a\}} x_b$ is $(t/N)^{k/2} H_{a,I}$, up to one irrelevant common sign. In the third sector, each exterior pair in J gives $(t/N)^{k/2} f_I$ multiplying the corresponding remaining monomial. Distinct added-variable monomials are Fischer-orthogonal and each has factorial weight one. The three exterior sectors are orthogonal. Taking squared norms proves (46). □

For clarity, this also identifies precisely what diagonal dilution adds to the derivative. Define

$$F(B)_{ij} = b_{ij} \operatorname{per} B(i \mid j), \qquad \operatorname{cut}_F(B[I]; a) = \sum_{\substack{i<a<j \\ i,j\in I}} F(B[I])_{ij},$$

where retained indices keep their original labels in the cut. The matrix $F(B[I])$ has row sums $\operatorname{per} B[I]$. Hence

$$\|H_{a,I}\|_{\mathrm{F}}^2 = |I| \operatorname{per} B[I] - 4 \operatorname{cut}_F(B[I]; a),$$

and (27) gives

$$D_1(A_N) = \sum_I \left(\frac{t}{N}\right)^{|I|} \left(D_1(B_N[I]) + 2 \sum_{a\in I^c} \operatorname{cut}_F(B_N[I]; a) \right). \tag{47}$$

The cross-block cuts are nonnegative, as follows by differentiating Lieb's cross-block scaling expansion, whose coefficients in the square of the scaling variable are nonnegative [10].

7.3 Conditional Bernoulli moments

For fixed z, let I be a random retained set with independent membership probabilities

$$p_i(z) = \frac{(t/N)|v_i \cdot z|^2}{1 + (t/N)|v_i \cdot z|^2}.$$

Summing $(t/N)^{|I|}|f_I(z)|^2$ produces exactly the product in (43). At $z = \sqrt{N}\, h$, these probabilities are $p_h(v_i)$.

Away from the zero sets of the nonzero linear forms, let

$$U_i = \xi_i \frac{v_i}{\ell_h(v_i)}, \qquad \xi_i \text{ independent Bernoulli}(p_h(v_i)).$$

Then

$$\mathbb{E}U_i = q_h(v_i), \qquad \mathbb{E}\|U_i\|^2 = \frac{t\|v_i\|^2}{1 + t|\ell_h(v_i)|^2} \le tL^2, \qquad \|\mathbb{E}U_i\| \le \frac{\sqrt{t}\, L}{2}. \tag{48}$$

These quotient random variables are used only almost everywhere. For each nonzero row its zero set is a complex hyperplane of Lebesgue measure zero. Zero rows contribute zero and can be omitted from this calculation. At a vanishing nonzero linear form, declaring $U_i = 0$ would not reproduce the continuous limiting second moment in (48); that identification is not made. Instead the moment expressions have continuous extensions in the original polynomial integrals, and values on these null sets do not affect an integral. The final functions $p_h, q_h, \mathcal{R}$ have no quotient singularities.

Set

$$W_N(h) = \frac{1}{N^2} \sum_{i<j} q_h(v_i) \wedge q_h(v_j), \qquad Q_N(h) = \frac{1}{N} \sum_i q_h(v_i),$$

$$Q_{a,N}(h) = \frac{1}{N} \sum_{i<a} q_h(v_i), \qquad m_N(h) = \frac{1}{N} \sum_i p_h(v_i).$$

For the first exterior sector the random quotient is

$$\frac{1}{N}\sum_{i<j} U_i \wedge U_j,$$

whose mean is NW_N. In its variance, disjoint pairs have zero covariance. There are $O(N^3)$ pairs sharing an index and $O(N^2)$ identical pairs. Every surviving covariance is bounded by a constant depending only on t, L, using independence and (48). The factor N^{-2} thus makes the variance $O(N)$. The contribution of this sector divided by $N^2 \operatorname{per} A_N$ equals

$$\mathbb{E}_{\Pi_N}\|W_N(h)\|^2 + O(N^{-1}). \tag{49}$$

For a fixed missing index a, the second-sector quotient is

$$\frac{1}{\sqrt{N}}\sum_{i\neq a} \operatorname{sign}(a-i)U_i.$$

Its mean is $\sqrt{N}(2Q_{a,N} - Q_N + q_h(v_a)/N)$, and its variance is at most tL^2. Membership of a is independent of this sum. Summing over a and using the uniform bounds gives the normalized second-sector contribution

$$\mathbb{E}_{\Pi_N}\frac{1}{N}\sum_a (1-p_h(v_a))\|Q_N(h) - 2Q_{a,N}(h)\|^2 + O(N^{-1}). \tag{50}$$

The third sector counts missing pairs. Its normalized contribution is

$$\frac{1}{2}\mathbb{E}_{\Pi_N}(1-m_N(h))^2 + O(N^{-1}). \tag{51}$$

All three error estimates are uniform in h.

7.4 Convergence of the ordered functionals

The discrete functions in (49)–(51) converge to their μ-integral versions locally uniformly in h. For the order indicators, partition the x-interval into short subintervals. The first marginal is Lebesgue measure, so boundaries $x = s$ and $x = y$ have zero μ and $\mu\otimes\mu$ mass. The uniform bound on q_h controls the contribution of the short subintervals. This proves prefix convergence uniformly in s. A finite h-net and uniform continuity give joint local uniformity in s, h; the same argument handles the ordered double integral.

Thus $\mathcal{R}$ is continuous and bounded, as are the discrete integrands uniformly. Tightness of Π_N, local uniform convergence, and the three sector estimates give

$$\frac{T(A_N)}{\binom{N}{2}\operatorname{per} A_N} = \mathbb{E}_{\Pi_N}\mathcal{R}(h) + o(1). \tag{52}$$

Concentration near K proves (35). Equation (28) proves the remaining assertions of Theorem 6.1.

8 Proof of the subcritical obstruction

8.1 A dominated Gaussian limit

Assume $t\lambda_{\max}(\Sigma) < 1$. Choose $s > 1$ and $b < 1$ such that $st\lambda_{\max}(\Sigma_N) \le b$ for all sufficiently large N. Since $\log(1+x) \le x$,

$$\prod_i \left(1 + \frac{st}{N}|v_i\cdot z|^2\right) \le e^{stz^*\Sigma_N z} \le e^{b\|z\|^2}.$$

This is integrable against the standard complex Gaussian, even after multiplication by any fixed power of $\|z\|$. The bounded row assumption and the quadratic Taylor remainder show pointwise convergence of the product with parameter t to $e^{tz^*\Sigma z}$. Dominated convergence gives

$$\operatorname{per} A_N \longrightarrow \mathbb{E}e^{tZ^*\Sigma Z} = \det(I - t\Sigma)^{-1},$$

which proves (40).

The positive principal-permanent weights

$$\mathbb{P}_N(I) = \frac{(t/N)^{|I|} \operatorname{per} B_N[I]}{\operatorname{per} A_N}$$

define a random retained size $K_N = |I|$. Its exponential moment is

$$\mathbb{E}s^{K_N} = \frac{\operatorname{per}(I + (st/N)B_N)}{\operatorname{per} A_N} \le C_s. \tag{53}$$

Therefore every fixed moment of K_N is uniformly bounded.

8.2 Polynomial bounds for compressed endpoint terms

For any $k \times k$ PSD Gram matrix B, put $C_k = \binom{k}{2}$. Cauchy–Schwarz applied to the C_k wedge summands gives

$$\begin{aligned} T(B) &\le C_k \sum_{i<j} \det B[\{i,j\}] \operatorname{per} B[\{i,j\}^c] \\ &\le C_k^2 \operatorname{per} B. \end{aligned} \tag{54}$$

The last inequality uses the elementary two-by-two inequality det $\le$ per, followed by Lieb's principal-block inequality [10]. The latter is also a consequence of the Fischer product inequality for products of linear forms [14]. It follows that

$$|D_1(B)| \le \frac{C_k + C_k^2}{2} \operatorname{per} B. \tag{55}$$

The matrix $F(B)$ is PSD: it is the Gram matrix of the vector-valued polynomials $v_i \prod_{j \ne i} \ell_j$. Lieb's inequality gives $F(B)_{ii} = b_{ii} \operatorname{per} B(i \mid i) \le \operatorname{per} B$. Thus $|F(B)_{ij}| \le \operatorname{per} B$, and

$$\left| \sum_{i,j} |i-j| F(B)_{ij} \right| \le k^3 \operatorname{per} B. \tag{56}$$

The polynomial dependence on k is needed here; a factorial bound would not suffice for (53).

8.3 Replacing inversions by the footrule

Define

$$\mathcal{L}(A) = \sum_{i,j} |i-j| a_{ij} \operatorname{per} A(i \mid j) = \sum_{\sigma} \left(\sum_i |i - \sigma(i)| \right) \prod_i a_{i,\sigma(i)}. \tag{57}$$

If a permutation fixes an index a, deleting that index reduces its inversion count by twice the number of edges crossing that cut in one direction. It reduces its footrule by the same amount, because

each crossing edge has its length reduced by one and the crossing counts in the two directions agree. Therefore

$$\operatorname{inv}(\sigma) - \sum_i |i - \sigma(i)|$$

is unchanged by deleting fixed indices and compressing the remaining order. Expanding $A_N = I + (t/N)B_N$ gives the exact identity

$$D_1(A_N) - \mathcal{L}(A_N) = \sum_I \left(\frac{t}{N}\right)^{|I|} (D_1(B_N[I]) - \mathcal{L}(B_N[I])), \tag{58}$$

where the submatrix quantities use their compressed order. Equations (53), (55), and (56) imply

$$|D_1(A_N) - \mathcal{L}(A_N)| \le C \operatorname{per} A_N \tag{59}$$

with C independent of N.

8.4 The limiting cross trace

For $i \neq j$, Gaussian expansion of the cross permanent gives

$$\operatorname{per} A_N(i \mid j) = \frac{t}{N}\, \mathbb{E}\left[(v_j \cdot Z)\overline{(v_i \cdot Z)} \prod_{a \neq i,j} \left(1 + \frac{t}{N}|v_a \cdot Z|^2\right)\right]. \tag{60}$$

The domination used above, with the required Gaussian moment weight, shows uniformly in i, j that the expectation in (60), divided by $\operatorname{per} A_N$, is $v_j^T R_0 v_i + o(1)$. Indeed, omitting two factors costs at most $O(N^{-1})\|Z\|^4$ times the dominating product; the full tilted second-moment matrix converges in the fixed dimension. Consequently

$$\frac{F(A_N)_{ij}}{\operatorname{per} A_N} = \frac{t^2}{N^2}(v_i^T v_j)(v_j^T R_0 v_i) + o(N^{-2})$$

uniformly for $i \neq j$. Weak convergence and (57) yield

$$\begin{aligned} \frac{\mathcal{L}(A_N)}{N \operatorname{per} A_N} &\longrightarrow t^2 \iint |x - y|(v^T w)(w^T R_0 v)\, d\mu(x, v)\, d\mu(y, w) \\ &= 2t^2 \int_0^1 \operatorname{tr}\big(M_s R_0(\Sigma - M_s)\big)\, ds. \end{aligned} \tag{61}$$

The last equality integrates the two cut orientations of $|x - y|$. The two traces agree and are real, since R_0 is a function of Σ.

Lemma 8.1 (A resolvent cross-trace inequality)**.** *If $A, B \succeq 0$, $S = A + B$, and $t\|S\| < 1$, then*

$$\operatorname{tr}\big(A(I - tS)^{-1}B\big) \ge \operatorname{tr}(AB) \ge 0. \tag{62}$$

Proof. Restrict to the range of S, since both A and B annihilate its kernel. Write $A = S^{1/2}CS^{1/2}$, $B = S^{1/2}(I - C)S^{1/2}$, with $0 \preceq C \preceq I$, and diagonalize S with eigenvalues $\lambda_i > 0$. If $f \ge 0$ and $x \mapsto xf(x)$ is nondecreasing on this spectrum, direct expansion gives

$$\begin{aligned} \operatorname{tr}(Af(S)B) = &\sum_i \lambda_i^2 f(\lambda_i)(C_{ii} - (C^2)_{ii}) \\ &+ \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)(\lambda_i f(\lambda_i) - \lambda_j f(\lambda_j))|C_{ij}|^2 \ge 0. \end{aligned} \tag{63}$$

Every term is nonnegative, because $C^2 \preceq C$. Apply this to $f(x) = (1-tx)^{-1} - 1$; then $xf(x) = tx^2/(1-tx)$ is nonnegative and increasing. This proves the first inequality in (62); the second is the standard trace inequality for two PSD matrices. □

The lower comparison is strictly positive after integration:

$$2\int_0^1 \operatorname{tr}\big(M_s(\Sigma - M_s)\big)\, ds = \iint |x-y||v^T w|^2 \, d\mu \, d\mu > 0. \tag{64}$$

The unweighted integral of $|v^T w|^2$ is $\operatorname{tr}(\Sigma^2) > 0$, while $x = y$ has zero $\mu \otimes \mu$ measure. More quantitatively, $|v^T w|^2 \le L^4$, and the mass of $|x-y| < \epsilon$ is at most 2ϵ. Taking $\epsilon = \operatorname{tr}(\Sigma^2)/(4L^4)$ gives

$$\iint |x-y||v^T w|^2 \, d\mu \, d\mu \ge \frac{(\operatorname{tr}\Sigma^2)^2}{8L^4}. \tag{65}$$

Equations (59) and (61), followed by Lemma 8.1, prove Theorem 6.3.

Proof of Corollary 6.4. The ordered empirical measures lie in a compact space of probability measures. Extract a weakly convergent subsequence from any proposed violating subsequence. Its first marginal is Lebesgue measure and its covariance is the corresponding limit of Σ_N. Apply Theorem 6.3 and (65) on that subsequence. This proves (42) and the version with uniform ε, s. □

9 The exact spherical saddles and a stronger Gini bound

9.1 The anisotropic spherical potential

Let Y be uniform on the real sphere S^2 and

$$v = T_\tau Y, \qquad T_\tau = \operatorname{diag}(1,1,\tau), \qquad 0 < \tau < 1.$$

The marginal law of v has covariance $\Sigma = \operatorname{diag}(1/3, 1/3, \tau^2/3)$. The order is specified below and does not affect Φ.

Proposition 9.1 (Exact coherent transition)**.** *For this spherical law, $h = 0$ is the unique maximizer of Φ when $t \le 3$. When $t > 3$, its maximizers are exactly*

$$h = e^{i\phi}\sqrt{m/2}(1,i,0) \quad \text{or} \quad h = e^{i\phi}\sqrt{m/2}(1,-i,0), \tag{66}$$

where $m \in (0,1)$ is uniquely determined by

$$m = m(k) := \int_0^1 \frac{k(1-z^2)}{1+k(1-z^2)}\, dz, \qquad k = \frac{tm}{2}. \tag{67}$$

Equivalently, $t = 2k/m(k)$, a strictly increasing function of $k > 0$, and

$$1 - m(k) = \frac{\operatorname{artanh}\sqrt{k/(1+k)}}{\sqrt{k(1+k)}}. \tag{68}$$

Proof. Multiply h by a phase so that the real and imaginary parts of $T_\tau h$ are orthogonal, and rotate Y. If their squared lengths are λ_1, λ_2, the logarithmic term in Φ becomes

$$\mathbb{E}\log(1 + t(\lambda_1 Y_1^2 + \lambda_2 Y_2^2)).$$

It is symmetric and strictly concave along each line with fixed positive $\lambda_1+\lambda_2$: the second derivative is the negative expectation of a nonzero square proportional to $(Y_1^2-Y_2^2)^2$. It is therefore maximized at $\lambda_1=\lambda_2$, and is strictly increasing in their sum. That sum is $\|T_\tau h\|^2 \le \|h\|^2$, with equality only if $h_3=0$. Thus, at fixed $\|h\|^2=m>0$, the maximizing directions are precisely (66).

The remaining radial function is strictly concave:

$$\varphi_t(m)=\int_0^1 \log(1+(tm/2)(1-z^2))\,dz-m.$$

Its derivative at zero is $t/3-1$. It tends to $-\infty$ as $m\to\infty$, so the asserted unique radial maximum follows. The stationary equation gives (67); its right side is strictly between zero and one. The function $m(k)/k$ is strictly decreasing, proving the asserted monotonicity of $t(k)$. Integrating $(1+k-kz^2)^{-1}$ gives (68). □

The transition at $t=3$ is a transition of the coherent maximizers. It does not by itself imply a negative endpoint derivative for every $t>3$; the ordered score must still exceed one.

9.2 A quantitative strict Gini inequality

Let R have density $1/[2(1+r^2)^{3/2}]$, independently of a uniform angle Θ. For an integrable real random variable Z, write $\mathcal{C}(Z)=\frac{1}{2}\mathbb{E}|Z-Z'|$, with an independent copy Z'. Put $X=\sqrt{2}R\cos\Theta$.

Lemma 9.2 (A stronger spherical Gini gain)**.** *One has*

$$\mathcal{C}(X)\ge\frac{1}{\sqrt{2}}+\frac{\pi^2-8}{32\sqrt{2}\,\pi}>\frac{1}{\sqrt{2}}+\frac{13}{1000}. \tag{69}$$

Proof. Let $U=|R|$ and $g(a,b)=\mathbb{E}\max(aU,bU')$ for $a,b\ge 0$. Independent symmetric signs show that $g(a,b)=\mathbb{E}|aR-bR'|$. The distribution function of U is $u/\sqrt{1+u^2}$, so

$$g(a,b)=\int_0^\infty\left(1-\frac{s^2}{\sqrt{(s^2+a^2)(s^2+b^2)}}\right)ds.$$

For positive a,b, differentiating the expectation of the maximum gives

$$g_{aa}(a,b)=\frac{1}{b}\int_0^\infty\frac{u^2\,du}{(1+u^2)^{3/2}(1+(a/b)^2u^2)^{3/2}}.$$

Differentiation is justified locally by an integrable majorant when a,b range in a compact subset of $(0,\infty)$. Homogeneity gives $g_{ab}=-(a/b)g_{aa}$ and $g_{bb}=(a/b)^2g_{aa}$.

Set $H(x)=g(1+x,1-x)$ and $a=1+x,b=1-x$. Substitution $u=s\sqrt{b/a}$ yields

$$H''(x)=\frac{4}{(ab)^{3/2}}\int_0^\infty\frac{s^2\,ds}{\big((1+s^2)^2+4(1-ab)s^2/(ab)\big)^{3/2}}.$$

Because $4s^2\le(1+s^2)^2$ and $0<ab\le 1$, the denominator inside the parentheses is at most $(1+s^2)^2/(ab)$. Hence

$$H''(x)\ge 4\int_0^\infty\frac{s^2}{(1+s^2)^3}\,ds=\frac{\pi}{4}.$$

Symmetry gives $H'(0)=0$. The radial distribution function $F_R(r)=\frac{1}{2}+r/(2\sqrt{1+r^2})$ gives

$$\mathcal{C}(R)=\int_{\mathbb{R}}F_R(r)(1-F_R(r))\,dr=\int_{\mathbb{R}}\frac{dr}{4(1+r^2)}=\frac{\pi}{4},$$

so $H(0) = \pi/2$. Integrating twice and rescaling proves

$$g(a,b) \geq \frac{\pi}{4}(a+b) + \frac{\pi}{16}\frac{(a-b)^2}{a+b}. \tag{70}$$

Continuity extends this to the boundary, interpreting the final term as zero at $a = b = 0$. For example, continuity follows from $\mathbb{E}U = 1$ and the Lipschitz bound for the maximum.

By symmetry X has the law of SR, with $S = \sqrt{2}|\cos\Theta|$ independent of R. Here

$$\mathbb{E}S = \frac{2\sqrt{2}}{\pi}, \qquad \mathbb{E}S^2 = 1, \qquad 0 \leq S \leq \sqrt{2}.$$

Average (70) over S, S'. Since $S + S' \leq 2\sqrt{2}$ and $\mathbb{E}(S - S')^2 = 2\,\mathrm{Var}(S)$, this gives

$$\mathcal{C}(X) = \tfrac{1}{2}\mathbb{E}g(S,S') \geq \frac{\pi}{4}\mathbb{E}S + \frac{\pi}{32\sqrt{2}}\,\mathrm{Var}(S) = \frac{1}{\sqrt{2}} + \frac{\pi^2 - 8}{32\sqrt{2}\,\pi}.$$

For the displayed rational bound, use $\pi > 157/50$, $\pi < 22/7$, and $\sqrt{2} < 99/70$. The resulting rational lower bound for the gain exceeds $13/1000$. □

10 A bounded-condition consequence

Use spherical coordinates

$$Y = (\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z),$$

where z is uniform on $[-1,1]$ and θ is independently uniform on $[0, 2\pi]$. Order rows by the real quotient

$$X_\tau(Y) = \frac{\sqrt{2}\,\tau z\cos\theta}{\sqrt{1-z^2}}. \tag{71}$$

Its values at the two sphere poles can be assigned arbitrarily, since those points have measure zero.

At the positive saddle in (66), write $a = \sqrt{m}$ and use the unitary coordinates with first basis vector $(1, i, 0)/\sqrt{2}$. Dividing a row form by its pole value gives

$$c = (1, e^{-2i\theta}, \sqrt{2}\,\tau z e^{-i\theta}/\sqrt{1-z^2}).$$

In these coordinates $q_h = (p/a)c$, with

$$p(z) = \frac{k(1-z^2)}{1 + k(1-z^2)}.$$

Consequently the real part of one exterior component is

$$C_{\text{eff}} := \Re(W_{02}) = \frac{1}{m} \cdot \frac{1}{2}\mathbb{E}[p(z)p(z')|X_\tau - X'_\tau|]. \tag{72}$$

Although the quotient is unbounded, $p\,c$ is bounded. Equivalently, q_h is a continuous bounded function on the original compact sphere. No logarithmic clipping or radial truncation is needed.

Lemma 10.1 (Loss from diagonal dilution)**.** *With $C_X = \mathcal{C}(\sqrt{2}R\cos\Theta)$ as above,*

$$C_{\text{eff}} \geq \tau\left(C_X - \frac{\sqrt{2}}{\sqrt{k}} - \frac{\log(1+k)}{k}\right). \tag{73}$$

Proof. Let $C_p = \frac{1}{2}\mathbb{E}[pp'|X_\tau - X'_\tau|]$. The inequality $1 - pp' \le (1-p) + (1-p')$, followed by the triangle inequality, gives

$$\tau C_X - C_p \le \mathbb{E}[(1-p)|X_\tau|] + (1-m)\mathbb{E}|X_\tau|.$$

Direct integration gives

$$\mathbb{E}|X_\tau| = \frac{2\sqrt{2}\,\tau}{\pi}, \qquad \mathbb{E}[(1-p)|X_\tau|] = \frac{2\sqrt{2}\,\tau}{\pi}\frac{\arctan\sqrt{k}}{\sqrt{k}}.$$

For the second identity substitute $u = \sqrt{1-z^2}$ into the radial integral. Also

$$1 - m = \int_0^1 \frac{dz}{1+k(1-z^2)} \le \int_0^1 \frac{dz}{1+k(1-z)} = \frac{\log(1+k)}{k}.$$

Now use $\arctan\sqrt{k} \le \pi/2$, $2\sqrt{2}/\pi < 1$, and $C_{\text{eff}} = C_p/m \ge C_p$. □

Theorem 10.2 (An eventual family with condition number below 20002)**.** *There is an explicit algebraic sequence of real positive-definite correlation matrices C_N, defined for every sufficiently large order, such that*

$$\limsup_{N\to\infty} \frac{D_1(C_N)}{N(N-1)\operatorname{per} C_N} < -\frac{1}{800}. \tag{74}$$

Its off-diagonal entries have modulus at most $60000/N$*, its least eigenvalue tends to one, and its condition number tends to* 20001*. In particular its condition number is eventually less than* 20002*. No explicit sufficient order is asserted in this theorem.*

Proof. Take

$$\tau = \frac{999}{1000}, \qquad t = 60000.$$

The saddle parameter satisfies $k > 25000$. Indeed $t(k) = 2k/m(k)$ is strictly increasing, and

$$1 - m(25000) \le \frac{\log 25001}{25000} < \frac{1}{6}$$

implies $t(25000) < 60000$. Both loss functions in (73) decrease with k. At $k = 25000$, the bounds $\sqrt{2} < 3/2$, $\sqrt{25000} > 158$, and $\log 25001 < 11$ give

$$\frac{\sqrt{2}}{\sqrt{k}} + \frac{\log(1+k)}{k} < \frac{3}{316} + \frac{11}{25000} < \frac{1}{100}.$$

For example, $\log 25001 < 11$ follows from $e > 8/3$ and $(8/3)^{11} > 25001$. Lemmas 9.2 and 10.1 therefore imply

$$C_{\text{eff}} > \tau\left(\frac{1}{\sqrt{2}} + \frac{3}{1000}\right) > \frac{1}{\sqrt{2}} + \frac{2}{1000}.$$

Using $\sqrt{2} > 5/4$ shows

$$2C_{\text{eff}}^2 > 1 + \frac{1}{200}.$$

The two saddle circles are conjugate, so their full score is constant. Theorem 6.1, even using only the first exterior sector, proves (74) for any ordered discretization converging to this sphere law.

Here is one deterministic algebraic discretization in every order. Let $m = \lfloor \sqrt{N} \rfloor$, and form the m^2 sphere rows

$$z_\ell = \frac{2\ell - 1}{m} - 1, \quad \theta_j = \frac{2\pi j}{m}, \quad Y_{j,\ell} = (\sqrt{1 - z_\ell^2}\cos\theta_j, \sqrt{1 - z_\ell^2}\sin\theta_j, z_\ell),$$

for $1 \le \ell \le m$, $0 \le j < m$. Apply T_τ, add $N - m^2$ copies of $(1, 0, 0)$, and sort by (71), breaking ties lexicographically by the original labels. The grid avoids the two poles and all entries are algebraic.

These empirical sphere measures converge to the uniform sphere law. The quotient is continuous away from the two poles and its limiting distribution has no atoms. Its empirical distribution functions therefore converge uniformly to its continuous distribution function F_X, and the largest relative tie multiplicity tends to zero. The sorted empirical position differs uniformly by $o(1)$ from $F_X(X_\tau(Y))$. Hence the joint ordered law converges to that of $(F_X(X_\tau(Y)), T_\tau Y)$. Its first marginal is uniform on $[0, 1]$, verifying (26).

Set $A_N = I + (60000/N)V_N V_N^T$ and normalize its diagonal:

$$C_N = \Delta_N^{-1/2} A_N \Delta_N^{-1/2}, \qquad \Delta_N = \operatorname{diag}(A_N).$$

Every permutation monomial, and hence the permanent and its derivative, is multiplied by the same positive factor. Thus the normalized derivative ratio is unchanged. The covariance eigenvalues converge to $1/3, 1/3, \tau^2/3$, so the condition number of A_N tends to $1 + 60000/3 = 20001$. Its least eigenvalue is one for large N. Since $\Delta_N = I + O(N^{-1})$ in operator norm and $\|A_N\|$ is bounded, $\|C_N - A_N\| \to 0$. This proves the asserted eigenvalue and condition-number limits. The unit bound on $\|T_\tau Y\|$, and diagonal normalization factors at least one, give the off-diagonal bound. □

Before correlation normalization, exactly $N - 3$ eigenvalues equal one when the rows span $\mathbb{R}^3$. Nonconstant diagonal normalization need not preserve that exact multiplicity; only the stated limiting spectral conclusions are claimed for C_N. This theorem concerns large orders only and supplies no explicit sufficient order; the finite certificate of Section 3 is a separate construction.

11 A four-dimensional sphere and condition numbers below 257

The second exterior sector can make a decisive contribution. We now use the complete score (34) for a four-dimensional real spherical law. The resulting condition-number bound is much smaller, while the preceding construction retains its advantage of using three-dimensional latent rows.

Let Y be uniform on $S^3 \subset \mathbb{R}^4$, and set

$$v = T_\tau Y, \qquad T_\tau = \operatorname{diag}(1, 1, \tau, \tau), \qquad 0 < \tau < 1.$$

Write

$$w_1 = Y_1 + iY_2, \qquad w_2 = Y_3 + iY_4, \qquad S = |w_1|^2, \qquad b = w_2/w_1.$$

The variable S is uniform on $[0, 1]$, and the two complex phases are independent and uniform. These statements follow, for example, by normalizing two independent standard complex Gaussians. The ratio b has density

$$\frac{1}{\pi(1 + |b|^2)^2}\, d^2 b. \tag{75}$$

Indeed $S = (1 + |b|^2)^{-1}$, and changing from the uniform S to the radius of b gives the radial density $2r/(1 + r^2)^2$. Order the rows by $x = \Re b$; the exceptional set $w_1 = 0$ has measure zero.

11.1 Exact saddle geometry and marginal formulas

The proof of Proposition 9.1 applies in real dimension four: rotational invariance and strict concavity in the two squared singular values force a maximizing complex direction to be circular in the first coordinate plane. For $t > 4$, the maximizers are exactly

$$h = e^{i\phi}\sqrt{m/2}(1, i, 0, 0) \quad \text{or its conjugate,}$$

where

$$m = \int_0^1 \frac{kS}{1+kS}\, dS = 1 - \frac{\log(1+k)}{k}, \qquad k = \frac{tm}{2}. \tag{76}$$

For completeness, the radial potential is $\int_0^1 \log(1 + (tm/2)S)\, dS - m$; it is strictly concave, has derivative $t/4 - 1$ at zero, and tends to $-\infty$. This proves uniqueness of the positive radial saddle. The function $t(k) = 2k/m(k)$ is strictly increasing by the same integral argument as before.

Put $K = 1 + k$, $\delta = 1 - m = \log K/k$, and

$$p = \frac{kS}{1+kS} = \frac{k}{K + |b|^2}.$$

The unweighted marginal density of $x = \Re b$, and its density weighted by p, are respectively

$$f(x) = \frac{1}{2(1+x^2)^{3/2}}, \tag{77}$$

$$f_p(x) = \frac{1}{2(1+x^2)^{3/2}} - \frac{1}{k\sqrt{1+x^2}} + \frac{1}{k\sqrt{K+x^2}}. \tag{78}$$

To check the second formula, use

$$\frac{k}{u^2(u+k)} = \frac{1}{u^2} - \frac{1}{ku} + \frac{1}{k(u+k)}$$

with $u = 1 + |b|^2$, and integrate the imaginary coordinate. Thus $f_p \geq 0$, $f - f_p \geq 0$, and their masses are m and δ, respectively.

Define the centered prefix mass and first moment

$$\mathcal{A}(x) = \int_{-\infty}^{x} f_p(u)\, du - \frac{m}{2}, \qquad \mathcal{M}(x) = \int_{-\infty}^{x} u f_p(u)\, du.$$

Elementary antiderivatives give

$$\mathcal{A}(x) = \frac{x}{2\sqrt{1+x^2}} - \frac{\operatorname{arsinh} x - \operatorname{arsinh}(x/\sqrt{K})}{k}, \tag{79}$$

$$\mathcal{M}(x) = -\frac{1}{2\sqrt{1+x^2}} + \frac{1}{\sqrt{K+x^2} + \sqrt{1+x^2}}. \tag{80}$$

For the second identity, the difference $(\sqrt{K+x^2} - \sqrt{1+x^2})/k$ is rationalized. The integration constant is fixed by the zero limit at $-\infty$.

11.2 The full score in one-dimensional integrals

Use the unitary basis $(1, \pm i, 0, 0)/\sqrt{2}$, $(0, 0, 1, \pm i)/\sqrt{2}$. If $\theta = \arg w_1$ and $a = e^{-2i\theta}$, the row divided by its first pole value has coefficients

$$c = (1, a, \tau b, \tau a\bar{b}). \tag{81}$$

The phase a is uniform and independent of b. Conditional on $x = \Re b$, the imaginary part of b has a symmetric law, also after multiplication by p. Consequently only coordinates zero and two of the averaged vector $q_h = (p/\sqrt{m})c$ survive, with densities

$$\frac{1}{\sqrt{m}}(f_p(x), 0, \tau x f_p(x), 0).$$

Writing

$$\Gamma_k = \frac{1}{2m} \iint |x - y| f_p(x) f_p(y)\, dx\, dy, \tag{82}$$

we obtain $\|W_h\|^2 = \tau^2 \Gamma_k^2$. The prefix vector gives

$$\|Q_h - 2Q_h(x)\|^2 = \frac{4}{m}(\mathcal{A}(x)^2 + \tau^2 \mathcal{M}(x)^2).$$

It follows that the exact limiting score at either saddle circle is

$$\mathcal{R}_* = 2\tau^2 \Gamma_k^2 + \frac{8}{m} \int_{\mathbb{R}} (f - f_p)(\mathcal{A}^2 + \tau^2 \mathcal{M}^2)\, dx + \delta^2. \tag{83}$$

All terms are finite. This calculation uses the complete Theorem 6.1; omitting the middle term would not give the bound below.

Lemma 11.1 (Bounds for the two nonconstant sectors)**.** *One has*

$$\Gamma_k \geq \frac{\pi/4 - 2\sqrt{K}/k}{m}. \tag{84}$$

If

$$\rho_k = \frac{1}{2} - \frac{\delta}{2} - \frac{1}{\sqrt{K} + 1} > 0,$$

then

$$\frac{8}{m} \int (f - f_p)(\mathcal{A}^2 + \tau^2 \mathcal{M}^2)\, dx \geq \frac{8\tau^2 \delta}{m} \rho_k^2. \tag{85}$$

Proof. For the probability density f, the absolute-difference potential is

$$g(x) = \int |x - y| f(y)\, dy = \sqrt{1 + x^2}.$$

Its derivative is $2F(x) - 1 = x/\sqrt{1 + x^2}$, and its value at zero is one, proving the formula. The half mean absolute difference of this law is $\pi/4$, by the radial calculation in Lemma 9.2. If $d = f - f_p$, expansion of $f_p = f - d$ yields

$$\frac{1}{2} \iint |x - y| f_p(x) f_p(y)\, dx\, dy \geq \frac{\pi}{4} - \int d(x) g(x)\, dx.$$

The omitted term is $\frac{1}{2}\iint |x-y|d(x)d(y)\,dx\,dy \geq 0$. Using (78),

$$\begin{aligned}\int d(x)g(x)\,dx &= \frac{1}{k}\int_{\mathbb{R}}\left(1-\frac{\sqrt{1+x^2}}{\sqrt{K+x^2}}\right)dx \\ &\leq \frac{2}{k}\int_0^\infty \left(1-\frac{x}{\sqrt{K+x^2}}\right)dx = \frac{2\sqrt{K}}{k}.\end{aligned}$$

Dividing by m proves (84).

For the mixed term, compare with the unweighted pair

$$\mathcal{A}_\infty(x) = \frac{x}{2\sqrt{1+x^2}}, \qquad \mathcal{M}_\infty(x) = -\frac{1}{2\sqrt{1+x^2}},$$

which satisfies $\mathcal{A}_\infty^2 + \mathcal{M}_\infty^2 = 1/4$. Equations (79)–(80) give the uniform estimates

$$|\mathcal{A}-\mathcal{A}_\infty| \leq \delta/2, \qquad |\mathcal{M}-\mathcal{M}_\infty| \leq \frac{1}{\sqrt{K}+1}.$$

The first follows also by integrating the even nonnegative density $f - f_p$ of mass δ from zero to x. The triangle inequality in $\mathbb{R}^2$ therefore gives

$$\mathcal{A}^2 + \tau^2\mathcal{M}^2 \geq \tau^2(\mathcal{A}^2+\mathcal{M}^2) \geq \tau^2\rho_k^2.$$

Integrating against $f - f_p$, whose mass is δ, proves (85). □

Theorem 11.2 (An eventual family with condition number below 257)**.** *There is an explicit algebraic sequence of real positive-definite correlation matrices E_N, defined for every sufficiently large order, such that*

$$\lim_{N\to\infty} \frac{D_1(E_N)}{N(N-1)\operatorname{per} E_N} < -\frac{1}{1000}. \tag{86}$$

Its off-diagonal entries have modulus at most $1020/N$, its least eigenvalue tends to one, and its condition number tends to 256. In particular its condition number is eventually below 257. The unnormalized mean-field perturbation has latent rank four. This theorem asserts no explicit sufficient order.

Proof. Set $\tau = 999/1000$ and $t = 1020$. Equation (76) implies $k < 510$. Since $m(500) > 1 - 7/500 > 50/51$, monotonicity of $t(k)$ gives $t(500) < 1020$, and therefore $k > 500$. On this interval,

$$6 < \log(1+k) < 7, \qquad \frac{6}{510} < \delta < \frac{7}{500}, \qquad 22 < \sqrt{K} < \frac{1131}{50}.$$

For example, the logarithmic bounds follow from $e < 11/4$, $(11/4)^6 < 501$, $e > 8/3$, and $(8/3)^7 > 511$. These are strict elementary inequalities. Using $\pi > 157/50$, Lemma 11.1 gives

$$\begin{aligned}\Gamma_k &> \frac{157/200 - 2(1131/50)/500}{1-6/510} = \frac{295171}{420000}, \\ \rho_k &> \frac{1}{2} - \frac{7}{1000} - \frac{1}{23} = \frac{10339}{23000} > 0.\end{aligned}$$

In (83), use $1/m > 1$ and $\delta^2 \geq \tau^2\delta^2$. The resulting entirely rational bound is

$$\begin{aligned}\mathcal{R}_* &> \left(\frac{999}{1000}\right)^2\left[2\left(\frac{295171}{420000}\right)^2 + 8\frac{6}{510}\left(\frac{10339}{23000}\right)^2 + \left(\frac{6}{510}\right)^2\right] \\ &= \frac{1505674035849200604129}{1498233800000000000000} > 1 + \frac{1}{250}.\end{aligned} \tag{87}$$

The full score is constant on the two conjugate saddle circles. Theorem 6.1 and (28) prove (86).

An explicit discretization uses $m = \lfloor N^{1/3} \rfloor$ and the m^3 rows

$$S_\ell = \frac{2\ell - 1}{2m}, \quad \theta_j = \frac{2\pi j}{m}, \quad \psi_a = \frac{2\pi a}{m},$$

$$Y_{j,a,\ell} = (\sqrt{S_\ell}\cos\theta_j, \sqrt{S_\ell}\sin\theta_j, \sqrt{1-S_\ell}\cos\psi_a, \sqrt{1-S_\ell}\sin\psi_a).$$

Here $1 \le \ell \le m$ and $0 \le j, a < m$. Apply T_τ, pad to N rows with $(1,0,0,0)$, and sort by

$$\Re b = \sqrt{\frac{1-S_\ell}{S_\ell}}\cos(\psi_a - \theta_j)$$

with a fixed lexicographic tie convention. All entries are algebraic and $w_1 \neq 0$ on the grid. The number of added rows is $O(N^{2/3})$. The unweighted empirical sphere measures converge to the uniform law. The limiting ordering coordinate has the continuous density f in (77); the same distribution-function argument used in Theorem 10.2 verifies ordered empirical convergence.

Set $A_N = I + (1020/N)V_N V_N^T$ and normalize its diagonal as before. Its covariance tends to $\mathrm{diag}(1/4, 1/4, \tau^2/4, \tau^2/4)$, so its condition number tends to $1 + 1020/4 = 256$. The row norms are at most one. Diagonal normalization preserves the derivative ratio, changes the operator by $o(1)$, and gives the claimed off-diagonal bound and spectral limits. □

Theorems 10.2 and 11.2 express a tradeoff: the first uses three-dimensional latent vectors and has the larger derivative margin, while the second uses four-dimensional latent vectors and has the much smaller condition-number bound. Both are eventual-order conclusions from conventional asymptotic proofs. Neither claims a finite negative Lean certificate.

11.3 A refinement to limiting condition number 227

A different choice of the two scalar parameters improves the conditioning bound using the same spherical law, ordering, and sector estimates.

Theorem 11.3 (An eventual family with condition number below 228)**.** *There is an explicit algebraic sequence of real positive-definite correlation matrices $\widetilde{E}_N$, defined in every sufficiently large order, such that*

$$\lim_{N\to\infty} \frac{D_1(\widetilde{E}_N)}{N(N-1)\,\mathrm{per}\,\widetilde{E}_N} < -\frac{1}{40000}, \qquad |(\widetilde{E}_N)_{ij}| \le \frac{904}{N} \quad (i \neq j).$$

Its least eigenvalue tends to one, and its condition number tends to 227, so it is eventually below 228. Before correlation normalization it is a mean-field perturbation with latent rank four. No explicit sufficient order is asserted.

Proof. Use the construction of Theorem 11.2, replacing its parameters by $\tau = 9999/10000$ and $t = 904$. Write $m(k) = 1 - \log(1+k)/k$ and $t(k) = 2k/m(k)$. The elementary bounds $5/2 < e < 11/4$, $(11/4)^6 < 445$, and $(5/2)^7 > 447$ give $6 < \log 445 < \log 447 < 7$. Consequently

$$t(444) < \frac{2\cdot 444^2}{437} = \frac{394272}{437} < 904, \qquad t(446) > \frac{2\cdot 446^2}{440} = \frac{49729}{55} > 904.$$

Strict monotonicity of $t(k)$ proves $444 < k < 446$. With $K = 1 + k$ and $\delta = 1 - m$, this implies

$$\frac{6}{446} < \delta < \frac{7}{444}, \qquad \frac{2109}{100} < \sqrt{K} < \frac{423}{20}.$$

The square-root inequalities follow by squaring: $(2109/100)^2 < 445$ and $(423/20)^2 > 447$. Using $\pi > 31415/10000$ in Lemma 11.1 gives

$$\Gamma_k > \frac{31415/40000 - 2(423/20)/444}{1 - 6/446} = \frac{45552433}{65120000} =: g > 0,$$
$$\rho_k > \frac{1}{2} - \frac{7}{888} - \frac{100}{2209} = \frac{876533}{1961592} =: r > 0.$$

As in (87), use $1/m > 1$ and $\delta^2 \geq \tau^2\delta^2$ in the full score. Exact rational arithmetic yields

$$\begin{aligned}\mathcal{R}_* &> \left(\frac{9999}{10000}\right)^2 \left[2g^2 + 8\frac{3}{223}r^2 + \left(\frac{3}{223}\right)^2\right] \\ &= \frac{42526916083267934083452500139764 1}{42522087382871168000000000000000000} > 1 + \frac{1}{10000}. \end{aligned} \tag{88}$$

Theorem 6.1 and (28) give the stated derivative limit.

Use the same algebraic sphere grid, padding, and sorting as in Theorem 11.2, with the new τ, and set $\widetilde{A}_N = I_N + (904/N)V_N V_N^T$. The row norms are at most one, and the limiting covariance is $\mathrm{diag}(1/4, 1/4, \tau^2/4, \tau^2/4)$. The condition number therefore tends to $1 + 904/4 = 227$. Correlation normalization preserves the normalized derivative, changes the operator by $o(1)$, and gives the off-diagonal bound $904/N$ and the least-eigenvalue limit as before. □

The family in Theorem 11.2 retains the larger proved derivative margin $1/1000$. The present theorem improves its condition-number bound with a smaller proved margin; it is a scalar refinement of the same analytic construction.

12 A spectral obstruction with an explicit positive margin

The full score in Theorem 6.1 admits an operator factorization that gives an obstruction in every fixed latent dimension. Throughout this section the rows and empirical measures satisfy the bounded-row hypotheses of Section 6. Eigenvalues of the limiting covariance Σ are listed in decreasing order, with missing eigenvalues interpreted as zero. All analytic arguments in this section are conventional proofs.

Theorem 12.1 (At most two supercritical covariance directions)**.** *Let $r \geq 1$, $t > 0$, and $\|v_{i,N}\| \leq L$, and suppose*

$$\Sigma_N = \frac{1}{N} V_N^T V_N \longrightarrow \Sigma, \qquad t\lambda_1(\Sigma) > 1, \qquad t\lambda_3(\Sigma) \leq 1.$$

For $A_N = I_N + (t/N)V_N V_N^T$, every sequence of row orders satisfies

$$\liminf_{N\to\infty} \frac{D_1(A_N)}{N(N-1)\operatorname{per} A_N} \geq \frac{\big(t\lambda_1(\Sigma) - 1\big)^2}{64r\, t^2 L^4} > 0. \tag{89}$$

In particular the endpoint derivative is eventually positive. The hypothesis includes the boundary $t\lambda_3(\Sigma) = 1$.

The nontrivial eigenvalues of A_N converge to $1 + t\lambda_j(\Sigma)$. Thus the theorem permits at most two limiting eigenvalues strictly above two, while requiring the largest one to exceed two. This is a statement about the specified bounded-row, fixed-rank mean-field families. Its conclusion does not assert a spectral criterion for unrestricted finite matrices.

12.1 The full score as a Hilbert–Schmidt norm

Fix an ordered limit μ as in (26) and a point $h \in \mathbb{C}^r$. Use $p = p_h$, $q = q_h$, and $m = m_h$ from (30)–(31), and define the real symmetric matrix

$$S_h = t \int \frac{vv^T}{1 + t|v \cdot h|^2} \, d\mu(x, v). \tag{90}$$

Let $H = L^2(\mu; \mathbb{C})$. On this space define

$$(\mathsf{K}f)(x, v) = \int \operatorname{sgn}(y - x) f(y, w) \, d\mu(y, w), \qquad \mathsf{Q}f = \int q f \, d\mu \in \mathbb{C}^r,$$

and set

$$\mathsf{D} = M_{\sqrt{1-p}}, \qquad \mathsf{T} : H \longrightarrow \mathbb{C}^r \oplus H, \qquad \mathsf{T}f = (\mathsf{Q}f, \mathsf{D}f).$$

For an operator between these complex Hilbert spaces, write $\mathsf{T}^{\mathrm{tr}} = J_H \mathsf{T}^* J_{\mathbb{C}^r \oplus H}$, where J is coordinatewise conjugation. This transpose has the same operator norm as T.

Lemma 12.2 (Exact factorization)**.** *The sign-kernel operator satisfies* $\|\mathsf{K}\|_{\mathrm{HS}}^2 = 1$*, and the full score* (34) *is*

$$\mathcal{R}(h) = \|\mathsf{T}\mathsf{K}\mathsf{T}^{\mathrm{tr}}\|_{\mathrm{HS}}^2. \tag{91}$$

If $S_h \preceq I_r$*, then* $\mathsf{T}^*\mathsf{T} \preceq I_H$*.*

Proof. The diagonal set $x = y$ has $\mu \otimes \mu$-measure zero because the position marginal is Lebesgue. Squaring the sign kernel therefore gives Hilbert–Schmidt norm squared one.

The old–old block $\mathsf{Q}\mathsf{K}\mathsf{Q}^{\mathrm{tr}}$ has entries

$$\iint \operatorname{sgn}(y - x) q_a(x, v) q_b(y, w) \, d\mu(x, v) d\mu(y, w).$$

It is the antisymmetric matrix of the ordered wedge W_h. Its matrix Hilbert–Schmidt norm squared is $2\|W_h\|^2$. Each of the two mixed blocks has squared norm

$$\int (1 - p(v)) \|2Q_h(x) - Q_h\|^2 \, d\mu(x, v).$$

The noise–noise block has kernel $\sqrt{1 - p(v)} \operatorname{sgn}(y - x) \sqrt{1 - p(w)}$, so its squared norm is $(1 - m)^2$. Adding the four block norms proves (91), including the factor two in the mixed term.

For the contraction assertion, put $b = q/\sqrt{p}$ on $\{p > 0\}$ and $b = 0$ elsewhere. Since $q = 0$ on $\{p = 0\}$,

$$\int bb^* \, d\mu = t \int_{\{p>0\}} (1 - p) vv^T \, d\mu \preceq S_h.$$

This is an inequality: rows with zero projection on h can still contribute to S_h. Weighted Cauchy–Schwarz consequently gives

$$\|\mathsf{Q}f\|^2 = \left\| \int b\sqrt{p} \, f \, d\mu \right\|^2 \leq \|S_h\| \int p|f|^2 \, d\mu \leq \int p|f|^2 \, d\mu.$$

Adding $\|\mathsf{D}f\|^2 = \int (1 - p)|f|^2 \, d\mu$ proves the claim. □

Lemma 12.3 (Quantitative contraction loss)**.** *If* $S_h \preceq I_r$*, then*

$$\mathcal{R}(h) \leq 1 - \frac{m_h^2}{16r}. \tag{92}$$

Proof. Put $\mathsf{P} = I_H - \mathsf{T}^*\mathsf{T} = M_p - \mathsf{Q}^*\mathsf{Q} \succeq 0$. The Hilbert–Schmidt ideal inequality and Lemma 12.2 imply

$$\mathcal{R}(h) \le \|\mathsf{TK}\|_{\mathrm{HS}}^2 = 1 - \operatorname{Tr}(\mathsf{K}^*\mathsf{PK}). \tag{93}$$

The conclusion follows immediately when $m = 0$, so assume $m > 0$. Choose an integer $d > r$ and write $w = m/d$. The cumulative p-mass is continuous, and $0 \le p \le 1$. Hence $[0, 1]$ can be partitioned into d consecutive intervals of p-mass exactly w, each of length at least w.

On interval j, let g_j rise linearly from zero to one over its first segment of length $w/4$, remain one, and fall linearly to zero over its last segment of length $w/4$. Extend it by zero and put $f_j = -g_j'/2$. Values at the finitely many corners do not matter. Integration and the Lebesgue position marginal give

$$\mathsf{K}f_j = g_j, \qquad \|f_j\|^2 = \frac{2}{w}, \qquad \int p|g_j|^2 \, d\mu \ge \frac{w}{2}. \tag{94}$$

For the last bound, discard the two edge strips. Their total length is $w/2$, so they contain at most $w/2$ of p-mass; the remaining mass lies on the plateau.

The disjoint supports make the f_j orthogonal. Their span E has dimension d, and (94) gives, for every $f \in E$,

$$\int p|\mathsf{K}f|^2 \, d\mu \ge \frac{w^2}{4}\|f\|^2.$$

The subspace $E \cap \ker(\mathsf{QK})$ has dimension at least $d - r$. Each unit vector in that subspace satisfies

$$\langle \mathsf{K}^*\mathsf{PK}f, f\rangle = \int p|\mathsf{K}f|^2 \, d\mu - \|\mathsf{QK}f\|^2 \ge \frac{w^2}{4}.$$

Summing over $d - r$ orthonormal vectors and using positivity of the trace-class operator gives

$$\operatorname{Tr}(\mathsf{K}^*\mathsf{PK}) \ge \frac{(d-r)m^2}{4d^2}.$$

Take $d = 2r$ in (93) to obtain (92). □

12.2 Stationarity and the third covariance eigenvalue

Lemma 12.4 (An inertia bound)**.** *Suppose h is a stationary point of Φ and $\operatorname{Re} h, \operatorname{Im} h$ are linearly independent. If S_h has u eigenvalues strictly greater than one, counted with multiplicity, then $t\Sigma$ has at least $u + 2$ eigenvalues strictly greater than one.*

Proof. Differentiation gives $S_h h = h$. Since S_h is real symmetric, it fixes the two-dimensional real space $E = \operatorname{span}_{\mathbb{R}}(\operatorname{Re} h, \operatorname{Im} h)$ pointwise. Let U be the sum of its eigenspaces with eigenvalue greater than one. Then $\dim U = u$ and $U \perp E$.

The covariance correction

$$C = t\Sigma - S_h = t\int p_h(v) v v^T \, d\mu \succeq 0 \tag{95}$$

is strictly positive on every nonzero vector $x \in E$. Indeed, $x^T S_h x = \|x\|^2 > 0$ implies positive measure for $\{v : v \cdot x \ne 0\}$. On this set $v \cdot h \ne 0$, since x is a real linear combination of the real and imaginary parts of h. Thus $p_h(v) > 0$ there and $x^T C x > 0$.

For a nonzero $x \in E \oplus U$, a nonzero U-component gives $x^T(S_h - I)x > 0$, while $x^T C x \ge 0$. If that component vanishes, $x \in E \setminus \{0\}$ and the strict positivity comes from C. Hence $t\Sigma - I$ is positive definite on the $(u+2)$-dimensional space $E \oplus U$. The min–max principle proves the claimed strict eigenvalue count. □

Lemma 12.5. *If $t\lambda_3(\Sigma) \le 1$, then $S_h \preceq I_r$ at every global maximizer h of Φ.*

Proof. If h is real up to a global phase, rotate that phase to make it real. For every real vector z, the imaginary variation gives

$$\frac{1}{2}\frac{d^2}{d\epsilon^2}\Phi(h+i\epsilon z)\bigg|_{\epsilon=0} = z^T(S_h - I)z \le 0.$$

Therefore $S_h \preceq I$; this also covers $h = 0$. Otherwise the real and imaginary parts are independent. An eigenvalue of S_h above one would, by Lemma 12.4, force $t\lambda_3(\Sigma) > 1$, a contradiction. □

Proof of Theorem 12.1. First suppose an ordered empirical limit exists. At every maximizer, Lemma 12.5 gives $S_h \preceq I$. For a real unit top eigenvector z of Σ,

$$1 \ge z^T S_h z = t\lambda_1(\Sigma) - t\int p_h(v)(v\cdot z)^2\, d\mu \ge t\lambda_1(\Sigma) - tL^2 m_h.$$

Consequently $m_h \ge (t\lambda_1(\Sigma) - 1)/(tL^2) > 0$, uniformly over the maximizing set. Lemma 12.3, Theorem 6.1, and the endpoint identity (28) now give (89).

For arbitrary changing orders, every subsequence has a further subsequence along which the ordered empirical measures converge weakly on the compact set $[0,1] \times \overline{B}(0,L)$. The limit has Lebesgue position marginal and covariance Σ. The lower bound just proved is the same for every such limit, because it depends only on $t, L, r, \lambda_1(\Sigma)$. A subsequence violating (89) would contradict its own ordered-limit bound. □

Corollary 12.6 (A necessary three-eigenvalue condition)**.** *Under the bounded-row and covariance-convergence assumptions, if*

$$\liminf_{N\to\infty} \frac{D_1(A_N)}{N(N-1)\operatorname{per} A_N} < 0,$$

then $t\lambda_3(\Sigma) > 1$. Thus a negative leading normalized endpoint derivative requires at least three limiting nontrivial eigenvalues of A_N strictly above two.

Proof. If $t\lambda_1(\Sigma) > 1$, this is the contrapositive of Theorem 12.1. If $t\lambda_1(\Sigma) \le 1$, then $\log(1+s) \le s$, with strict inequality for $s > 0$, shows $\Phi(h) < 0$ for every $h \ne 0$. If the row projection vanishes almost everywhere, the same conclusion follows from $\Phi(h) = -\|h\|^2$. Thus the maximizing set is $\{0\}$ and $\mathcal{R}(0) = 1$. Theorem 6.1 gives convergence of the displayed normalized derivative to zero, including the first critical boundary $t\lambda_1(\Sigma) = 1$. Compactness of ordered empirical subsequences gives the same conclusion for changing orders. □

Corollary 12.7 (Latent rank at most two)**.** *For real rows in $\mathbb{R}^2$, if $t\lambda_1(\Sigma) > 1$, then every sequence of orders satisfies*

$$\liminf_{N\to\infty} \frac{D_1(A_N)}{N(N-1)\operatorname{per} A_N} \ge \frac{\big(t\lambda_1(\Sigma) - 1\big)^2}{128t^2L^4} > 0.$$

Together with Theorem 6.3 and Corollary 6.4, this proves eventual endpoint increase in both noncritical regimes when $\Sigma \ne 0$.

12.3 Scope and the rank-two literature

The first critical boundary can still leave derivatives of smaller order than N^2 per A_N; the sign of such derivatives is not decided here. A public manuscript of liqsweep [11] claims permanent-on-top for all real PSD matrices of rank at most two. If that claim is accepted, its hereditary form, closure under nonnegative diagonal addition, and the endpoint sign-Gram identity already imply finite-order endpoint nonnegativity for that class plus a nonnegative diagonal. The proofs above use no result from that manuscript. They give an explicit strict asymptotic margin and apply in arbitrary latent dimension through the number of supercritical covariance eigenvalues.

13 Exponential rarity of negative endpoint orders

The product-limit calculation in Corollary 6.2 has a probabilistic consequence for uniformly permuted deterministic row clouds. It uses standard bounded-difference concentration in addition to the full mean-field criterion.

Corollary 13.1 (Uniform random orders)**.** *Fix $t > 0$, $r \geq 1$, and deterministic rows $v_{1,N}, \dots, v_{N,N} \in \overline{B}(0, L)$. Suppose*

$$\nu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{v_{j,N}} \Longrightarrow \nu, \qquad \Sigma = \int v v^T \, d\nu(v), \qquad t\lambda_{\max}(\Sigma) > 1.$$

Let π_N be a uniformly distributed permutation of the N row labels, and set

$$A_N^{\pi_N} = I_N + (t/N) V_N^{\pi_N} (V_N^{\pi_N})^T.$$

There exist constants $\eta, c, C > 0$ and N_0 such that, for $N \geq N_0$,

$$\mathbb{P}\left\{ \frac{D_1(A_N^{\pi_N})}{N(N-1) \operatorname{per} A_N^{\pi_N}} < \eta \right\} \leq C e^{-cN}. \tag{96}$$

In particular the fraction of labelled permutations with nonpositive endpoint derivative is at most Ce^{-cN}. Under any joint coupling of the uniform permutations across orders, the displayed normalized derivative is at least η almost surely for all sufficiently large N. Independence across orders is not required.

Proof. Set $\rho = dx \, \nu(dv)$. For this product ordered limit, Corollary 6.2 gives the strictly positive number

$$c_* = \min_{h \in K_\rho} \frac{m_h(4 - m_h)}{12} > 0.$$

Take $\eta = c_*/2$. There is a weak neighborhood U of ρ and an order N_1 such that every bounded-row array of order $N \geq N_1$ with ordered empirical measure in U has normalized derivative at least η. Otherwise, choose successively smaller neighborhoods of ρ and increasing orders to obtain empirical measures converging to ρ, each with normalized derivative below $c_*/2$. This contradicts the liminf bound c_* from Theorem 6.1 and Corollary 6.2.

By the definition of the weak topology on the compact row-position space, choose finitely many real continuous functions $f_1, \dots, f_J$, $\epsilon > 0$, and $B = \max(1, \max_j \|f_j\|_\infty)$ such that

$$\left\{ \mu : \left| \int f_j \, d\mu - \int f_j \, d\rho \right| < \epsilon \text{ for } 1 \leq j \leq J \right\} \subseteq U.$$

For a uniform permutation π, define

$$X_{j,N} = \frac{1}{N} \sum_{i=1}^{N} f_j(i/N, v_{\pi(i),N}).$$

Its expectation is

$$\mathbb{E}X_{j,N} = \frac{1}{N^2} \sum_{i,a=1}^{N} f_j(i/N, v_{a,N}) \longrightarrow \int f_j \, d\rho.$$

Indeed the deterministic position-grid measures converge to Lebesgue measure and $\nu_N \Longrightarrow \nu$, so their product measures converge weakly. For large enough N, every expectation above is within $\epsilon/2$ of its limit.

Swapping two assigned row labels changes $X_{j,N}$ by at most $4B/N$. Reveal the permutation one position at a time. Conditional on the already exposed labels, completions for any two choices of the next label can be coupled by exchanging just those two labels. Thus the conditional expectations for the two choices differ by at most $4B/N$, and each increment of the associated Doob martingale has absolute value at most $4B/N$.

The bounded-increment martingale inequality of Azuma [1] now gives

$$\mathbb{P}\{|X_{j,N} - \mathbb{E}X_{j,N}| \geq \epsilon/2\} \leq 2 \exp\left(-\frac{\epsilon^2 N}{128B^2}\right). \tag{97}$$

For clarity, this version follows directly by applying conditional convexity to a centered increment Z with $|Z| \leq a$: $\mathbb{E}(e^{sZ} \mid \mathcal{F}) \leq \cosh(sa) \leq e^{s^2a^2/2}$. Iteration and Markov's inequality give the two-sided bound $2e^{-u^2/(2Na^2)}$; substitute $a = 4B/N$ and $u = \epsilon/2$.

The event in (96) forces the ordered empirical measure outside U. The union bound in (97) proves the assertion with $C = 2J$, $c = \epsilon^2/(128B^2)$, and a sufficiently large N_0. Finally $\sum_N Ce^{-cN} < \infty$, so the first Borel–Cantelli lemma gives the almost-sure conclusion under any joint coupling. □

The constants and the sufficient order in this corollary are existential: the weak-neighborhood argument does not supply explicit finite rates for the mean-field approximation. When a deterministic cloud admits an ordered endpoint counterexample, the corollary shows that such negative endpoint orders occupy an exponentially small fraction of its labelled permutations. It makes no assertion of monotonicity over the full q-interval for the other orders.

14 Formal proof and reproducibility

14.1 What the Lean development proves

The accompanying Lean project pins Lean 4.19.0 and Mathlib revision

`c44e0c8ee63ca166450922a373c7409c5d26b00b`.

It has 32 proof modules. A separate audit module enumerates all 2068 declarations whose defining module is one of them, including private and generated declarations, and computes their transitive axioms. Of these, 1319 are auxiliary declarations that the Lean 4.19 code generator stores for executable definitions; beyond the standard axioms they may refer only to its placeholder `lcProof` and to one another, and no other declaration depends on them through an axiom. Each of the other 749 declarations is a theorem, definition or inductive construction, not an axiom, and depends only on

{`propext`, `Classical.choice`, `Quot.sound`}.

There is no `sorry`, no added axiom and no native evaluation; in particular `Lean.ofReduceBool` does not occur.

The development defines the inversion-weighted polynomial P_q of an ordered matrix by its permutation sum, exactly as in Section 2, and proves that D_q is its derivative. It proves, for every real matrix V with three columns, the identity $2D_1(VV^T) = \binom{n}{2}\|f\|_{\mathrm{F}}^2 - \|G\|_{\mathrm{F}}^2$ of Lemma 2.1, with both Fischer norms written as factorial-weighted coefficient sums of the actual product and wedge polynomials. It proves the four-row recurrence of Lemma 3.1, the parity decomposition (9)–(11), the semantics of an executable sparse integer implementation of the recurrence, and certificate soundness: for every list L of integer types, if the integer test $T_U > \binom{n}{2}P$ computed from L succeeds, then the Gram matrix of the row word of L has $D_1 < 0$, and there are a rational matrix of the same order that is positive definite and rationals $0 < q_1 < q_2 < 1$ with $P_{q_2} < P_{q_1}$. The last step is a general reduction: a real positive-semidefinite matrix with negative endpoint derivative yields a rational positive-definite matrix and a rational interior decreasing pair. Identity padding is proved for the actual polynomial, together with preservation of positive definiteness and rationality; one fixed rational pair therefore serves every larger order.

For the frozen input, the development defines the 896 ordered rows from the reversed type list, proves that the Gram matrix is positive semidefinite, and proves that the row matrix and the Gram matrix have rank exactly three. The rank proof uses the rows with zero-based indices $0, 1, 894$, whose matrix is

$$\begin{pmatrix} -3 & 2 & 20 \\ -3 & -2 & 20 \\ 3 & 2 & 20 \end{pmatrix}, \qquad \det = 480,$$

through a lemma about the outermost type that avoids expanding all 224 nested row constructors. This is a different minor from the one in Table 1; both certify the same rank. Finally it proves that each of the following hypotheses implies the all-orders statement

$$\exists q_1, q_2 \in \mathbb{Q},\ 0 < q_1 < q_2 < 1, \quad \forall N \geq 896\ \exists A_N \in \mathbb{Q}^{N\times N}, \quad A_N \succ 0, \quad P_{q_2}(A_N) < P_{q_1}(A_N) :$$

the integer test above for the frozen type list, and the Fischer-norm inequality $\binom{896}{2}\|f\|_{\mathrm{F}}^2 < \|G\|_{\mathrm{F}}^2$ for the frozen rows.

The project also contains scalar lemmas used by the asymptotic sections: bounds for the spherical Gini gain, logarithmic and radial errors, rational score margins and condition-number arithmetic for the families of Theorems 10.2, 11.2 and 11.3, and two finite matrix steps of Section 12. These lemmas formalize the indicated inequalities only; they do not formalize the probabilistic constructions or limits that supply their hypotheses.

14.2 What is checked outside Lean

The evaluation of the integer test on the frozen input is not carried out in Lean. Kernel reduction of the 224-step recurrence, whose polynomials have up to about 10^5 coefficients of several thousand digits, is far beyond practical limits. An earlier stage of this work evaluated it with Lean's native code generator through `native_decide`, which adds `Lean.ofReduceBool` and hence the compiler and runtime to the trusted base [9]; that module is not part of the present development. Instead, the hypothesis of the conditional theorems is checked by the two exact programs of Section 3: the integer implementation of the four-row recurrence, and the independent row-by-row modular computation, which determines P, T_U and T through the Chinese remainder theorem. The finite counterexample is therefore a formal reduction together with an exact computation; it is not a closed Lean theorem.

The explicit matrix $A = MB + I_n$ and the explicit pair $q_1 = 1 - 2/Q$, $q_2 = 1 - 1/Q$ of Section 4 have conventional proofs; the formal positive-definite conclusion is existential. The

mean-field endpoint functional, Gaussian concentration, subcritical asymptotic, spherical saddle analysis, algebraic grids, operator factorization, spectral obstruction, random-order theorem and coherent-ordering theorem are conventional proofs as well. Their finite matrix and scalar ingredients are formalized only where stated above.

15 Limits and further questions

The finite certificate provides an economical explicit real witness, but it does not establish the minimum possible order. The formal rank-three construction does not prove that rank two is impossible at every parameter; no minimum-rank theorem is asserted here. Monotonicity on the whole negative half-interval and failure at an arbitrarily prescribed positive interior parameter are not settled here.

The mean-field result distinguishes a coherent-saddle threshold from the sign of the endpoint derivative. The general limiting functional contains three nonnegative exterior contributions, and negativity of D_1 requires their combined score to exceed one. The subcritical formula shows that this cannot happen below limiting condition number two within the stated fixed-rank class. The gap between that obstruction and the explicit upper construction at 227 remains open. The stronger spectral obstruction also requires three limiting nontrivial eigenvalues above two for a strictly negative N^2-scale derivative. It does not resolve the sign of a vanishing normalized derivative at the first critical threshold, and is not a theorem about every finite positive-definite matrix satisfying a spectral bound. The asymptotic theorems do not supply a numerical sufficient order for their particular well-conditioned families. The finite matrix and the bounded-condition constructions should not be conflated.

16 Research process and AI disclosure

This is fully autonomous agentic research. The finite witness, its exact certificate and four-row recurrence, the regularization, the asymptotic and spectral theory, the original Lean development (for Lean 4.19.0), an evaluation of the certificate with native code, and the literature search were produced by autonomous AI research agents in earlier sessions. Their records do not name the system used. The audits and separate implementations in those records were produced within the same automated workflow.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It reran the integer recurrence program on the frozen inputs, wrote and ran the independent row-by-row modular computation and the Chinese-remainder determination of Section 3, checked the endpoint identity and the regularization of Section 4 line by line, removed the native evaluation from the formal development, added the frozen-input module with its rank and conditional all-orders theorems, ran the fresh Lean build and whole-project axiom audit, and revised this manuscript. The asymptotic and spectral sections were carried over from the earlier sessions and were read, but not rederived line by line, in this release.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed; no expert validation is claimed, AI checks are not human peer review, and the novelty assessment is provisional and does not establish priority. Section 14 states the formal scope.

A Complete ordered integer data

The following table is read by increasing index. Each triple is a type (a, b, c), added as the outer four-row operation in Section 3. Thus the list is inner-to-outer. In a list implementation that recursively wraps the first type around the tail, the input must be reversed; the formal source records this conversion explicitly.

Table 2: All 224 four-row types, in inner-to-outer order.

j	(a,b,c)	j	(a,b,c)	j	(a,b,c)	j	(a,b,c)
1	$(0,1,0)$	2	$(0,1,0)$	3	$(0,1,0)$	4	$(0,1,0)$
5	$(0,1,0)$	6	$(0,1,0)$	7	$(0,1,0)$	8	$(0,1,0)$
9	$(0,1,0)$	10	$(0,1,0)$	11	$(0,1,0)$	12	$(0,1,0)$
13	$(0,1,0)$	14	$(0,1,0)$	15	$(1,23,11)$	16	$(1,23,11)$
17	$(1,30,20)$	18	$(19,24,2)$	19	$(19,24,2)$	20	$(31,32,3)$
21	$(31,32,3)$	22	$(11,25,2)$	23	$(11,25,2)$	24	$(12,31,5)$
25	$(15,28,3)$	26	$(15,28,3)$	27	$(18,28,3)$	28	$(18,28,3)$
29	$(29,16,2)$	30	$(29,16,2)$	31	$(20,9,1)$	32	$(30,25,3)$
33	$(19,16,2)$	34	$(32,22,3)$	35	$(32,22,3)$	36	$(3,23,10)$
37	$(3,23,10)$	38	$(6,29,10)$	39	$(14,5,1)$	40	$(14,5,1)$
41	$(31,7,2)$	42	$(31,7,2)$	43	$(18,2,1)$	44	$(28,3,2)$
45	$(29,1,1)$	46	$(2,23,23)$	47	$(1,11,11)$	48	$(2,24,31)$
49	$(2,24,31)$	50	$(7,21,6)$	51	$(7,21,6)$	52	$(13,29,7)$
53	$(4,27,17)$	54	$(4,27,17)$	55	$(3,27,22)$	56	$(3,27,22)$
57	$(9,15,4)$	58	$(9,15,4)$	59	$(23,31,8)$	60	$(7,29,16)$
61	$(7,29,16)$	62	$(6,19,8)$	63	$(6,19,8)$	64	$(5,11,4)$
65	$(5,11,4)$	66	$(10,17,6)$	67	$(30,31,9)$	68	$(30,31,9)$
69	$(17,21,7)$	70	$(17,21,7)$	71	$(25,21,7)$	72	$(31,26,9)$
73	$(16,11,4)$	74	$(6,25,20)$	75	$(6,25,20)$	76	$(8,21,12)$
77	$(8,21,12)$	78	$(26,15,6)$	79	$(26,15,6)$	80	$(19,9,4)$
81	$(19,7,4)$	82	$(19,7,4)$	83	$(29,7,6)$	84	$(29,7,6)$
85	$(16,31,16)$	86	$(16,31,16)$	87	$(4,12,9)$	88	$(14,2,3)$
89	$(14,2,3)$	90	$(24,1,5)$	91	$(24,1,5)$	92	$(4,6,3)$
93	$(19,28,14)$	94	$(19,24,12)$	95	$(20,21,11)$	96	$(20,21,11)$
97	$(28,24,13)$	98	$(28,24,13)$	99	$(6,23,25)$	100	$(6,23,25)$
101	$(10,29,27)$	102	$(20,32,21)$	103	$(20,32,21)$	104	$(14,30,23)$
105	$(14,30,23)$	106	$(32,20,13)$	107	$(32,20,13)$	108	$(4,3,2)$
109	$(31,12,11)$	110	$(31,12,11)$	111	$(27,14,11)$	112	$(29,15,12)$
113	$(1,7,12)$	114	$(1,7,12)$	115	$(5,21,29)$	116	$(30,6,11)$
117	$(30,6,11)$	118	$(10,3,4)$	119	$(10,3,4)$	120	$(28,32,23)$
121	$(25,29,21)$	122	$(21,19,14)$	123	$(20,1,8)$	124	$(20,1,8)$
125	$(31,3,13)$	126	$(31,3,13)$	127	$(15,26,24)$	128	$(15,26,24)$
129	$(13,28,28)$	130	$(24,25,22)$	131	$(24,25,22)$	132	$(9,12,11)$
133	$(9,12,11)$	134	$(17,13,12)$	135	$(17,13,12)$	136	$(31,19,20)$
137	$(25,10,14)$	138	$(25,10,14)$	139	$(27,14,17)$	140	$(25,13,16)$
141	$(8,20,27)$	142	$(9,22,30)$	143	$(12,25,30)$	144	$(16,3,9)$

Continued on the next page

Table 2 continued

j	(a,b,c)	j	(a,b,c)	j	(a,b,c)	j	(a,b,c)
145	$(16,3,9)$	146	$(24,8,15)$	147	$(27,9,17)$	148	$(32,25,29)$
149	$(23,18,21)$	150	$(23,22,26)$	151	$(17,20,24)$	152	$(17,20,24)$
153	$(15,23,29)$	154	$(12,18,23)$	155	$(1,5,11)$	156	$(3,14,31)$
157	$(6,17,30)$	158	$(13,1,9)$	159	$(13,1,9)$	160	$(18,2,13)$
161	$(18,2,13)$	162	$(11,6,10)$	163	$(11,6,10)$	164	$(22,17,25)$
165	$(8,3,7)$	166	$(8,3,7)$	167	$(16,7,15)$	168	$(16,7,15)$
169	$(9,9,14)$	170	$(16,16,25)$	171	$(17,12,21)$	172	$(13,19,31)$
173	$(7,10,17)$	174	$(9,17,31)$	175	$(7,13,24)$	176	$(20,15,29)$
177	$(21,16,31)$	178	$(9,9,17)$	179	$(31,1,32)$	180	$(30,1,31)$
181	$(29,1,31)$	182	$(28,1,30)$	183	$(15,1,17)$	184	$(15,1,17)$
185	$(18,1,21)$	186	$(19,2,23)$	187	$(19,2,23)$	188	$(12,1,15)$
189	$(12,1,15)$	190	$(3,10,29)$	191	$(3,10,29)$	192	$(5,10,24)$
193	$(13,12,29)$	194	$(12,11,27)$	195	$(3,4,10)$	196	$(7,9,23)$
197	$(9,0,14)$	198	$(16,0,25)$	199	$(8,0,13)$	200	$(13,0,22)$
201	$(17,0,29)$	202	$(9,0,16)$	203	$(5,0,9)$	204	$(8,0,15)$
205	$(10,0,19)$	206	$(1,0,2)$	207	$(12,0,25)$	208	$(9,0,19)$
209	$(14,0,31)$	210	$(4,0,9)$	211	$(11,0,26)$	212	$(5,0,12)$
213	$(11,0,28)$	214	$(3,6,25)$	215	$(4,7,31)$	216	$(5,6,28)$
217	$(5,6,29)$	218	$(3,0,10)$	219	$(9,0,31)$	220	$(4,0,15)$
221	$(6,0,25)$	222	$(5,0,22)$	223	$(3,2,19)$	224	$(3,2,20)$

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