Exact three-uniform Berge-path saturation numbers and a counterexample to a conjectured formula

Dev D. Goyal

31 October 2026; revised 1 November 2026

Abstract

For $m \geq 10$ write $m = 4s + r$ with $1 \leq r \leq 4$, and put $d = m - 2$, $a = (3 + r)2^s - 5$ and $b = 2a + 1$. English, Graber, Kirkpatrick, Methuku and Sullivan conjectured [1] that the three-uniform saturation number of the Berge path P_m equals $U(n) = \lceil (n - \lfloor n/b \rfloor)/2 \rceil$ for every $n \geq b$. Two disjoint loose 8-cycles together with two isolated vertices form a Berge-P_{10}-saturated three-uniform hypergraph on 34 vertices with 16 hyperedges, one fewer than the conjectured value 17. We determine the saturation number exactly on the whole conjectured range: for all $m \geq 10$ and $n \geq b$, $\mathrm{sat}_3(n, \text{Berge-}P_m) = U(n) - 1$ when $b < n < 2b$ and $n \equiv 2 \pmod{2d}$, and $\mathrm{sat}_3(n, \text{Berge-}P_m) = U(n)$ otherwise. At every exceptional order the extremal hypergraph is unique: disjoint loose d-cycles and two isolated vertices. The lower bound uses two results of the source paper, the minimum size of a saturated linear tree and the general upper construction. Lean 4 verifies, for actual hypergraphs, Berge paths and unrestricted saturation, that disjoint loose d-cycles with two isolated vertices are Berge-P_{d+2}-saturated for every $d \geq 3$, that the explicit 34-vertex hypergraph is Berge-P_{10}-saturated with 16 hyperedges, and that the three-uniform case of the conjecture fails for every $m \geq 10$. The matching lower bound and the uniqueness statement have conventional proofs.

1 The conjecture and the result

All hypergraphs are finite and three-uniform; the vertex set includes isolated vertices. A *Berge path of length L* in a hypergraph H consists of distinct vertices $v_0, \ldots, v_L$ and distinct hyperedges $e_1, \ldots, e_L$ of H with $\{v_{i-1}, v_i\} \subseteq e_i$ for every i. The graph path P_m has m vertices, so a Berge-P_m is a Berge path of length $m-1$. A hypergraph H is *Berge-P_m-saturated* if it contains no Berge-P_m but $H + f$ contains one for every three-element set $f \notin E(H)$; no condition such as linearity is imposed on f. The saturation number $\mathrm{sat}_3(n, \text{Berge-}P_m)$ is the least number of hyperedges of a Berge-P_m-saturated hypergraph on n vertices [1].

A *linear tree* is a connected hypergraph whose incidence graph is a tree; it has $2e + 1$ vertices when it has e hyperedges. Fix $m \geq 10$, write $m = 4s + r$ with $1 \leq r \leq 4$ and put

$$d = m - 2, \qquad a = (3 + r)2^s - 5, \qquad b = 2a + 1, \qquad U(n) = \left\lceil \frac{n - \lfloor n/b \rfloor}{2} \right\rceil .$$

We use two results of [1] for $k = 3$.

(T) (Theorem 2.2.) Every Berge-P_m-saturated linear tree on at least four vertices has at least a hyperedges, and one with exactly a hyperedges exists.

(C) (Construction 3.8, Observation 3.9 and Lemma 3.10.) For every $n \geq b$ there is a Berge-P_m-saturated hypergraph on n vertices with $U(n)$ hyperedges.

Theorem 2.3 of [1] gives a lower bound that differs from $U(n)$ by at most one, and Conjecture 2.4 of [1] asserts, for $k = 3$, that $\mathrm{sat}_3(n, \text{Berge-}P_m) = U(n)$ for all $m \geq 10$ and $n \geq b$. The authors note that the bounds would agree if a minimal saturated hypergraph could be shown to have no isolated vertices. Isolated vertices are exactly what breaks the formula.

Theorem 1. *Let $m \geq 10$ and $n \geq b$. Then*

$$\mathrm{sat}_3(n, \textit{Berge-}P_m) = \begin{cases} U(n) - 1, & \textit{if } b < n < 2b \textit{ and } n \equiv 2 \pmod{2d}, \\ U(n), & \textit{otherwise.} \end{cases}$$

At an exceptional order $n = 2td + 2$ every extremal hypergraph is isomorphic to t disjoint loose d-cycles together with two isolated vertices.

Corollary 2. *The three-uniform case of Conjecture 2.4 of [1] is false for every $m \geq 10$. For $m = 10$ and $n = 34$ the conjectured value is 17, while two disjoint loose 8-cycles with two isolated vertices give a Berge-P_{10}-saturated hypergraph with 16 hyperedges.*

The novel parts are the exceptional construction, the treatment of isolated vertices and the resulting matching lower bound; results (T) and (C) are inherited. Corollary 2 needs only Proposition 3 below and arithmetic, and it is verified in Lean (Section 7).

2 Loose cycles with two isolated vertices

For $d \geq 3$, a *loose d-cycle* has core vertices $c_0, \ldots, c_{d-1}$, pendant vertices $p_0, \ldots, p_{d-1}$ and hyperedges $e_i = \{c_i, c_{i+1}, p_i\}$, indices modulo d. Let $\mathcal{C}_{d,t}$ consist of t disjoint loose d-cycles and two isolated vertices z_1, z_2; it has $2td + 2$ vertices and td hyperedges.

Proposition 3. *For $d \geq 3$ and $t \geq 0$, $\mathcal{C}_{d,t}$ is Berge-P_{d+2}-saturated.*

Proof. A Berge path is connected through its hyperedges, so all its hyperedges lie in one component; each component has at most d hyperedges, while a Berge-P_{d+2} needs $d + 1$. Hence $\mathcal{C}_{d,t}$ is Berge-P_{d+2}-free.

Every vertex of a cycle starts a Berge path through all d hyperedges of its cycle: from c_i use $c_i, c_{i+1}, \ldots, c_{i+d-1}, p_{i+d-1}$ with $e_i, \ldots, e_{i+d-1}$, and from p_i use the reverse of the path $c_{i+1}, \ldots, c_{i+d}, p_{i+d} = p_i$. Let f be an absent triple.

If f contains a vertex u of some cycle and a vertex w outside that cycle, prepend w, f to the d-edge path from u. Since there are only two isolated vertices, this covers every f that is not contained in a single cycle.

If f lies in one cycle, two of its vertices have the same type. If $p_i, p_j \in f$ with $i \neq j$, prepend p_j, f to the d-edge path from p_i, which contains no other pendant vertex. If $c_x, c_y \in f$, let $k \in \{1, \ldots, d-1\}$ with $y \equiv x + k$. Take

$$p_{x+k-1},\ c_{x+k-1},\ c_{x+k-2}, \ldots, c_x,\quad c_y,\ c_{y+1}, \ldots, c_{y+d-k-1},\ p_{y+d-k-1}$$

with the hyperedges $e_{x+k-1}, \ldots, e_x$, then f, then $e_y, \ldots, e_{y+d-k-1}$. The two runs of cores occupy the disjoint offset ranges $[0, k-1]$ and $[k, d-1]$ from x, the pendants p_{x+k-1} and p_{x-1} are distinct, and every old hyperedge occurs once. This is a Berge path with $k + 1 + (d - k) = d + 1$ hyperedges. □

3 Incidence counting and isolated hyperedges

For a component with v vertices and e hyperedges, the incidence graph has $v + e$ nodes and $3e$ edges, hence cyclomatic number $2e - v + 1$. A linear tree has $v = 2e + 1$; every other component has $v \leq 2e$, with equality exactly for incidence-unicyclic components. Berge paths correspond to paths of the incidence graph between vertex nodes.

Let H be Berge-P_m-saturated on $n \geq b$ vertices. It has at most two isolated vertices, since a triple of isolated vertices could be added without creating a path of length $d + 1 \geq 9$.

Lemma 4. *If H has a component consisting of one hyperedge, then $|E(H)| \geq \lceil (n-1)/2 \rceil \geq U(n)$.*

Proof. Let $\{x, y, w\}$ be that component and suppose that T is another linear tree component, possibly a single vertex. Since H is Berge-P_m-free, Berge paths in T have length at most d, so its incidence tree has diameter at most $2d$. A centre of this tree, replaced by an adjacent vertex node if it is a hyperedge node, gives a vertex v of T from which every Berge path in T has length at most $\lceil d/2 \rceil$. A Berge path in $H + \{x, y, v\}$ that uses the new hyperedge uses at most one hyperedge of $\{x, y, w\}$ and at most $\lceil d/2 \rceil$ of T, so it has length at most $\lceil d/2 \rceil + 2 \leq d$, contradicting saturation. Hence every other component has $v \leq 2e$, and $|E(H)| \geq 1 + (n-3)/2$. Since $\lfloor n/b \rfloor \geq 1$, the bound is at least $U(n)$. □

4 One isolated vertex

Lemma 5. *Let z be an isolated vertex of H, let T be a linear tree component with at least one hyperedge and let $e = \{x, y, w\}$ be a hyperedge of T. Removing e splits T into three branches containing x, y and w; let h_x, h_y, h_w be the largest lengths of Berge paths in these branches starting at x, y, w, labelled so that $h_x \geq h_y \geq h_w$. Then $h_y = h_w$ and $h_x + h_y = d - 1$.*

Proof. Joining longest paths in the x- and y-branches through e gives a path of length $h_x + h_y + 1$, so $h_x + h_y \leq d - 1$. Add $f = \{z, x, y\}$. A new path uses f. If it avoids e, it either joins the x- and y-branches through f or ends at z, and has length at most $h_x + h_y + 1 \leq d$. If it uses e and f, these must be consecutive with a common vertex $c \in \{x, y\}$ between them, because the branches are joined only through e and f. The remaining ends are then one of w or the other element of $\{x, y\}$ on the side of e, and one of z or the other element of $\{x, y\}$ on the side of f; checking the four possibilities gives length at most $h_x + h_w + 2$. Saturation forces $h_x + h_w \geq d - 1$, and with $h_w \leq h_y$ both equalities follow. □

In particular every such T has Berge diameter exactly d.

Lemma 6. *Under the hypotheses of Lemma 5, T has at least B_m hyperedges, where $B_m = 2(2^h - 1)$ if $d = 2h$ and $B_m = 3 \cdot 2^h - 2$ if $d = 2h + 1$. Moreover $B_m \geq 2a$ for all $m \geq 10$.*

Proof. The incidence tree has diameter $2d$, and its centre is a vertex node when d is even and a hyperedge node when d is odd.

Let $d = 2h$ with centre c. Root the tree at c; at least two branches at c reach Berge depth h, and no vertex has depth more than h. For a hyperedge whose parent vertex has depth j, the parent side has height exactly $h + j$ (through another deep branch at c), and both child sides have height at most $h - j - 1$. Lemma 5 forces both child heights to equal $h - j - 1$. Hence every child vertex of depth less than h has a child hyperedge, and a deep branch at c contains at least $1 + 2 + \cdots + 2^{h-1} = 2^h - 1$ hyperedges. Two deep branches give $2(2^h - 1)$.

Let $d = 2h + 1$ with central hyperedge e_0. Its three branches have height at most h, and Lemma 5 applied to e_0 makes all three equal to h. Below a vertex of e_0, a hyperedge whose parent is j steps from that vertex has parent side of height $h + j + 1$ and child sides of height at most $h - j - 1$, which Lemma 5 again makes equal. Each branch therefore has at least $2^h - 1$ hyperedges, and T has at least $1 + 3(2^h - 1)$.

For $m = 10, 11, 12, 13$ the values (a, B_m) are $(15, 30)$, $(19, 46)$, $(23, 62)$ and $(27, 94)$. Moreover $a_{m+4} = 2a_m + 5$ and $B_{m+4} = 4B_m + 6$, so $B_m \geq 2a_m$ implies $B_{m+4} - 2a_{m+4} = 4(B_m - a_m - 1) \geq 0$. □

Corollary 7. *If H has exactly one isolated vertex, then $|E(H)| \geq U(n)$.*

Proof. By Lemma 4 we may assume that no component is a single hyperedge. Let t be the number of linear tree components with a hyperedge; by Lemma 6 each has at least $2B_m + 1 \geq 2b - 1$ vertices, so $n \geq 1 + t(2b - 1) \geq (t + 1)b$ when $t \geq 1$, and $\lfloor n/b \rfloor \geq t + 1$ in all cases. Counting vertices over the non-isolated components gives $2|E(H)| \geq n - 1 - t \geq n - \lfloor n/b \rfloor$. □

5 Two isolated vertices

Lemma 8. *If H has two isolated vertices z_1, z_2, then every non-isolated vertex v starts a Berge path of length d, and no vertex node is a cut node of the incidence graph of its component.*

Proof. A path created by adding $\{z_1, z_2, v\}$ must end with this hyperedge at an isolated vertex, so it consists of a path of length d from v and one more step. If a vertex node v were a cut node, a longest path from v would lie in one part after deleting v, and a hyperedge at v in another part, together with one of its other vertices, would extend it to length $d+1$. □

Lemma 9. *Under the hypotheses of Lemma 8, every incidence-unicyclic component is a loose d-cycle.*

Proof. Let Q be the cycle of the incidence graph. A hyperedge node off Q would lie in a tree hanging from Q, and the first vertex node on the way from it to the rest of the component, which is either the attachment node or the node following a hyperedge attachment node, would be a cut node separating two parts that contain hyperedges. So every hyperedge lies on Q; its third vertex is a pendant vertex, since a chord would create a second cycle, and every vertex on Q has degree two. The component is a loose cycle with L hyperedges, its longest Berge paths have length L, and every vertex starts one; Lemma 8 gives $L = d$. □

With two isolated vertices there is no linear tree component with a hyperedge: one with at least two hyperedges has a vertex cut node, and a single hyperedge has longest path length one. Hence $|E(H)| \geq \lceil (n-2)/2 \rceil$, and if n is even and equality holds then every non-isolated component is incidence-unicyclic, hence a loose d-cycle, and $n = 2td + 2$.

6 Proof of Theorem 1

Let H be Berge-P_m-saturated on $n \geq b$ vertices and $q = \lfloor n/b \rfloor$. If a component is a single hyperedge, Lemma 4 applies. With no isolated vertex, each linear tree component is itself Berge-P_m-saturated on its vertex set and has at least four vertices, so by (T) it has at least a hyperedges and at least b vertices; there are at most q of them, and counting gives $2|E(H)| \geq n - q$. With one isolated vertex, Corollary 7 applies. With two isolated vertices $|E(H)| \geq \lceil (n-2)/2 \rceil$, which is at least $U(n)$ when $q \geq 2$ and equals $U(n)$ when $q = 1$ and n is odd. If $q = 1$ and n is even, so that $b < n < 2b$, then $U(n) = n/2$, and $|E(H)| = n/2 - 1$ forces disjoint loose d-cycles and two isolated vertices, hence $n \equiv 2 \pmod{2d}$ and the stated uniqueness. Conversely $\mathcal{C}_{d,t}$ attains $n/2 - 1$ at $n = 2td + 2$ by Proposition 3, and (C) supplies $U(n)$ hyperedges at every other order. □

Exceptional orders exist for every $m \geq 10$: one checks $a \geq d$, and $t = \lceil a/d \rceil$ gives $a \leq td \leq 2a - 1$, so $b < 2td + 2 < 2b$. They are the orders $2td + 2$ with $\lceil a/d \rceil \leq t \leq \lfloor (2a-1)/d \rfloor$.

m	d	a	b	exceptional n	$\mathrm{sat}_3(n, \text{Berge-}P_m)$ versus $U(n)$
10	8	15	31	34, 50	16, 24 versus 17, 25
11	9	19	39	56, 74	27, 36 versus 28, 37
12	10	23	47	62, 82	30, 40 versus 31, 41
13	11	27	55	68, 90	33, 44 versus 34, 45

7 Formal verification

The Lean 4.35.0 project in `lean/` uses Mathlib revision `56a005855ea17e0b0958f73ec801ae140b0a1679`. A hypergraph is a `Finset` of vertex sets; `IsBergePath` requires distinct vertices, distinct hyperedges of the hypergraph and

the linking condition above; `ContainsBergePath H m` asks for one of length $m-1$; and `IsBergePathSaturated` requires three-uniformity, freeness and a Berge-P_m after adding any absent three-element set. The saturation number `satNumber n m` is the infimum of the sizes of saturated hypergraphs on `Fin n`, and `Conjecture24ThreeUniform` states the conjectured equality with a and $U(n)$ written as above. The project proves:

- `looseCycles_saturated`: Proposition 3 for every $d \geq 3$ and every t;
- `witness34_saturated` and `witness34_card`: the hypergraph on `Fin 34` with the triples $\{16j+i,\ 16j+((i+1) \bmod 8),\ 16j+8+i\}$, $j \in \{0,1\}$, $0 \leq i \leq 7$, is Berge-P_{10}-saturated and has 16 hyperedges, while `conjecturedSat 10 34 = 17`;
- `not_conjecture24ThreeUniform`, `satNumber_lt_conjecturedSat` and `conjecture24_fails_for_every_m`: the conjecture fails, the saturation number is below $U(n)$ at every order $2t(m-2)+2$ strictly between b and $2b$, and every $m \geq 10$ has such an order.

A module audit lists the transitive axioms of all 191 declarations of the three proof modules, including generated ones; all depend only on `propext`, `Classical.choice` and `Quot.sound`. The lower bound of Theorem 1, the uniqueness statement and the inherited results (T) and (C) are not formalized. A standard-library Python program checks Proposition 3 independently by exhaustive search for $(m,n) = (10,34)$, $(10,50)$ and $(11,56)$.

8 Prior work and limitations

The definitions, results (T) and (C), the lower bound of Theorem 2.3 and Conjecture 2.4 are from the arXiv version of [1]; the journal version [2] could not be accessed for this release, so whether its wording of the conjecture differs has not been checked. Linearity of Berge saturation numbers for small uniformities is due to English, Gerbner, Methuku and Tait [3], and graph path saturation to Kászonyi and Tuza [4]. Searches in October 2026 of arXiv, journal records and general web indexes did not find the loose-cycle construction, a counterexample to Conjecture 2.4 or an exact formula; this does not establish priority. Theorem 1 covers only uniformity three, $m \geq 10$ and $n \geq b$.

9 Research process and AI disclosure

This is fully autonomous agentic research. The question selection, the counterexample, the proof of Theorem 1, the literature search and exhaustive computer checks were produced by autonomous AI research agents in an earlier session. Its records attribute the work, including an internal AI review that rederived the formula and audited the proof, to an Astra agent workflow, and they state that no Lean formalization was attempted.

Opus 5.5 by Anthropic used with Claude Code assessed the package for this release. It read the relevant statements of [1], checked the written proof, rechecked the witnesses with its own exhaustive search, wrote and ran the Lean formalization described in Section 7, and wrote this note. The editorial revision of 1 November 2026, also made with Opus 5.5 by Anthropic used with Claude Code, changed the presentation only, not the mathematics or the formalization.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed; no expert validation is claimed, AI checks are not human peer review, and the novelty assessment is provisional. Section 7 states the formal scope, and Section 8 the remaining limits.

References

[1] S. English, N. Graber, P. Kirkpatrick, A. Methuku and E. C. Sullivan, *Saturation of Berge hypergraphs*, arXiv:1710.03735v1 (2017).

[2] S. English, P. Gordon, N. Graber, A. Methuku and E. C. Sullivan, *Saturation of Berge hypergraphs*, Discrete Mathematics **342** (2019), 1738–1761. doi:10.1016/j.disc.2019.01.031.

[3] S. English, D. Gerbner, A. Methuku and M. Tait, *Linearity of saturation for Berge hypergraphs*, European Journal of Combinatorics **78** (2019), 205–213. arXiv:1807.06947.

[4] L. Kászonyi and Zs. Tuza, *Saturated graphs with minimal number of edges*, Journal of Graph Theory **10** (1986), 203–210.