Exact conditional contraction for Pauli channels and depolarization in every dimension

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Abstract

This paper determines the conditional trace-distance contraction coefficient, in the framework of Hirche, George, Nuradha and Wilde [1], of every stochastic Pauli qubit channel and of the full physical depolarizing family in every dimension. If a, b are the two smallest Pauli probabilities, the answer is $1 - 8ab/(a+b)$, with value one when both vanish. For $d \geq 2$ and negative depolarization $\mathcal{N}_s(X) = (1+s)\operatorname{Tr}(X)I_d/d - sX$, $0 \leq s \leq 1/(d^2-1)$, the answer is $s[1+(d^2-1)s]/[1-(d^2-3)s]$. The positive-multiplier branch has its standard value λ. All conditional hockey-stick parameters $\gamma \geq 1$ have these same values. The negative-parameter upper bound holds for every Hermitian input with every finite quantum reference; a balanced tight frame encoded in a genuine Bell basis supplies matching normalized density witnesses. An explicit nonlinear decomposition of binary effects proves the upper bound without asserting a physical degrading channel. The resulting sharp binary-comparison threshold can be of order d^{-2} while the physical erasure-simulation threshold remains constant. A Lean 4 formalization verifies the exact coefficients for actual finite-reference density states, every stochastic Pauli qubit channel and the full depolarizing family; the existence of the unital-qubit normal form is cited, and the operational corollaries have conventional proofs (Section 9).

1 The research problem and its scope

The quantity studied here measures how much distinguishability survives a channel after the distinguishability already present in a retained quantum reference is subtracted. Its optimization includes unequal reference marginals and arbitrarily large finite references. Computing an ordinary trace contraction coefficient, or imposing equal reference marginals, does not solve this optimization.

The framework and general positive-Doeblin upper bound are established in Hirche, George, Nuradha and Wilde [1], building on the quantum Doeblin theory of George, Hirche, Nuradha and Wilde [2]. The present problem is quantitative: determine the exact coefficient, with an unrestricted upper bound and matching actual states, for the entire Pauli family and for depolarization in every dimension. Theorems below give full parameter ranges, all ties and boundary cases, rather than a numerical estimate or a selected special parameter.

The proof combines a reference-flag identity, a sharp qubit effect bound and an all-dimensional matching lower witness with its arbitrary-effect upper bound. Harmonic interpolation, Bell bases, standard qubit channel normal forms and the cited support-vector constructions are credited ingredients. Section 8 compares the result with prior work. The Lean formalization covers Theorems 1 and 2 for actual density states and finite references; the unital-qubit normal form is cited, and the corollaries of Section 7 are conventional proofs (Section 9).

Earlier public work (note added 1 November 2026). On 31 October 2026 a check of public problem records found that the Omega Institute's public repository records a claimed answer to the question in Section 9 of [1] (issue #17637), committed on 16 October 2026, before the first version of this manuscript [8]. It claims that, for every channel between finite systems, the conditional hockey-stick coefficient is optimized by orthogonal states and equals the conditional

trace-distance coefficient for every $\gamma \geq 1$. That general theorem covers the γ-independence of the formulas here (Section 3); it does not compute the exact Pauli and depolarizing values. No originality or priority claim is made for the γ-independence.

2 Definitions and exact statements

Let $\mathcal{N} : A \to B$ be a finite-dimensional CPTP map and $\mathcal{N}_R = \mathrm{id}_R \otimes \mathcal{N}$. For density operators define

$$T(\rho, \sigma) = \tfrac{1}{2} \|\rho - \sigma\|_1, \qquad E_\gamma(\rho\|\sigma) = \mathrm{Tr}(\rho - \gamma\sigma)_+, \qquad \gamma \geq 1.$$

Write $\rho_R = \mathrm{Tr}_A \rho$ and

$$\Delta_\gamma(\rho, \sigma) = E_\gamma(\rho\|\sigma) - E_\gamma(\rho_R\|\sigma_R).$$

The coefficient is the literal supremum

$$\eta^{\Delta}_{E_\gamma}(\mathcal{N}) = \sup_{\substack{R,\rho,\sigma \\ \Delta_\gamma(\rho,\sigma)>0}} \frac{\Delta_\gamma(\mathcal{N}_R\rho, \mathcal{N}_R\sigma)}{\Delta_\gamma(\rho, \sigma)}. \tag{1}$$

All finite R and all PSD trace-one input states are admitted. At $\gamma = 1$ this is η^{Δ}_T. For comparison,

$$\eta^{c}_T(\mathcal{N}) = \sup_{\substack{R,\rho,\sigma:\rho_R=\sigma_R \\ T(\rho,\sigma)>0}} \frac{T(\mathcal{N}_R\rho, \mathcal{N}_R\sigma)}{T(\rho, \sigma)}.$$

The positive Doeblin coefficient uses the unnormalized Choi matrix $J_{\mathcal{N}}$:

$$\alpha_+(\mathcal{N}) = \sup\{\mathrm{Tr}\, P : P \succeq 0,\ J_{\mathcal{N}} \succeq I_A \otimes P\}.$$

For Hermitian X set $X_R = \mathrm{Tr}_A X$ and

$$g(X) = \mathrm{Tr}\, X_+ - \mathrm{Tr}(X_R)_+ = \tfrac{1}{2}(\|X\|_1 - \|X_R\|_1). \tag{2}$$

Thus $\Delta_\gamma(\rho, \sigma) = g(\rho - \gamma\sigma)$. A channel preserves X_R, a fact used explicitly in the formal development.

Theorem 1 (Every Pauli channel)**.** *Let $w_0, w_1, w_2, w_3 \geq 0$ sum to one and*

$$\mathcal{P}_w(X) = \sum_{j=0}^{3} w_j \sigma_j X \sigma_j^\dagger, \qquad (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (I, X, Y, Z).$$

Let $a \leq b$ be the two smallest weights, counted with multiplicity. For every $\gamma \geq 1$,

$$\eta^{\Delta}_{E_\gamma}(\mathcal{P}_w) = \eta^{\Delta}_T(\mathcal{P}_w) = 1 - \frac{8ab}{a+b}. \tag{3}$$

If $a = b = 0$, the right side means one. The positive Doeblin coefficient is $4a$.

The deficit in (3) is the harmonic mean of $4a$ and $4b$. The standard unitary normal form for unital qubit channels [3, Theorem 2.2], together with invariance under input and output unitary conjugations, gives the corresponding mathematical formula for every unital qubit channel. In terms of the two least eigenvalues μ_1, μ_2 of its *unnormalized* Choi matrix, it is $1 - 4\mu_1\mu_2/(\mu_1 + \mu_2)$. The supplied Lean theorem directly covers all stochastic Pauli channels; normal-form existence itself is not encoded in this project.

Theorem 2 (Depolarization in every dimension)**.** *For $d \geq 2$ let*

$$\mathcal{D}_\lambda(X) = \lambda X + (1-\lambda)\operatorname{Tr}(X) I_d/d, \qquad -\frac{1}{d^2-1} \leq \lambda \leq 1.$$

For every $\gamma \geq 1$,

$$\eta^{\Delta}_{E_\gamma}(\mathcal{D}_\lambda) = \eta^{\Delta}_{T}(\mathcal{D}_\lambda) = \begin{cases} \lambda, & 0 \leq \lambda \leq 1, \\ \dfrac{s[1+(d^2-1)s]}{1-(d^2-3)s}, & \lambda = -s, \quad 0 \leq s \leq 1/(d^2-1). \end{cases} \tag{4}$$

For the negative branch, put $n = d^2$ and $p_n(s) = s[1+(n-1)s]/[1-(n-3)s]$. Then

$$\eta^{c}_{T}(\mathcal{N}_s) = s, \qquad \alpha_+(\mathcal{N}_s) = 1-(n-1)s.$$

In the strict physical interior,

$$s < p_n(s) < (n-1)s.$$

The denominator is positive throughout the closed interval: it equals $1-(n-1)s+2s$, and at the upper endpoint it is $2/(n-1)$. The formal negative-branch theorem uses the explicit canonical d-dimensional basis indexed by $\mathbb{Z}/d\mathbb{Z}$; d is arbitrary. This is a theorem in every dimension, not an assertion that arbitrary index-type reindexing has separately been formalized.

3 Why all conditional hockey-stick parameters agree

The flag argument fixes the precise meaning of the γ extension. Add one orthogonal basis vector to the *reference* and choose any density matrix ω on A. Given ρ, σ and $\gamma \geq 1$, set

$$P = \gamma^{-1}\rho \oplus (1-\gamma^{-1})\omega, \qquad Q = \sigma \oplus 0.$$

Then $P - Q = \gamma^{-1}(\rho - \gamma\sigma) \oplus (1-\gamma^{-1})\omega$. A positive block supported on the added reference flag contributes exactly its trace to both full and reference positive traces, so it cancels from g. Both the input and output gaps are scaled by γ^{-1}. Conversely, set

$$P = \rho \oplus 0, \qquad Q = \gamma^{-1}\sigma \oplus (1-\gamma^{-1})\omega.$$

Now $P - \gamma Q = (\rho - \sigma) \oplus [-(\gamma-1)\omega]$ and the negative added block contributes zero to both positive traces. Both gaps are unchanged. Positive denominators are preserved, so the actual attainable ratio sets coincide, not merely their suprema. It can enlarge the reference and change purity; no fixed pure-reference claim at $\gamma > 1$ is made.

4 The sharp upper bound for negative depolarization

Assume initially $0 < s < 1/(n-1)$, $n = d^2$, and let $0 \preceq M \preceq I$ be an arbitrary effect on $R \otimes A$. Define

$$S = d\,\operatorname{Tr}_A(M) \otimes I_A, \quad b = \frac{1+s}{n}, \quad t = \frac{ns}{1+s}, \quad f(x) = \frac{(1+s)x}{ns + [1-(n-1)s]x}.$$

With the unnormalized vector $|\Omega\rangle = \sum_{j=0}^{d-1} |j,j\rangle$, the completely positive map $X \mapsto d\operatorname{Tr}(X)I - X$ has Choi matrix $dI - |\Omega\rangle\langle\Omega| \succeq 0$, since $\||\Omega\|\|^2 = d$. Applying this map with the full reference to M and $I - M$ yields

$$0 \preceq S \preceq nI, \qquad M \preceq S, \qquad I - M \preceq nI - S. \tag{5}$$

This use of complete positivity is essential: no commutation between M and the reference algebra is assumed.

4.1 An explicit harmonic matrix certificate

Here is the entire noncommutative interpolation step. If $0 \preceq M \preceq I, S$, let

$$C = [tI + (1-t)S]^{-1}, \qquad H = SC = f(S).$$

The inverse exists because $0 < t < 1$ and $S \succeq 0$. One has $tC+(1-t)H = I$ and $tCSC+(1-t)H^2 = H$. Direct expansion gives

$$\begin{aligned} H - M = {} & tC(S-M)C + (1-t)H(I-M)H \\ & + t(1-t)(C-H)M(C-H) \succeq 0. \end{aligned} \tag{6}$$

All three terms are positive semidefinite sandwiches. Apply the same identity to $I - M$ and $nI - S$ in (5) to obtain

$$I - f(nI - S) \preceq M \preceq f(S).$$

The formal proof uses spectral functions and verifies (6) algebraically, including singular S; it does not assume an operator-monotonicity principle.

4.2 The scalar inequality that fixes the constant

Write

$$q = \frac{1+(n-1)s}{1-(n-3)s}, \quad p = sq, \quad D(x) = ns + [1-(n-1)s]x.$$

For $0 \le x \le n$, exact algebra gives

$$1 + q - f(x) - f(n-x) = \frac{s(1+s)[1-(n-1)s](2x-n)^2}{[1-(n-3)s]D(x)D(n-x)} \ge 0. \tag{7}$$

Also $sf(x) \le bx$. Define the scalar function

$$z(x) = \max\left\{ f(x), \frac{bx-(1-p)}{s} \right\}, \qquad h(x) = bx - sz(x).$$

The preceding two bounds, applied also to $n - x$, imply

$$f(x) \le z(x) \le 1 - f(n-x) + q, \qquad 0 \le h(x) \le 1 - p.$$

For the second branch of the maximum, the upper bound follows from $sf(n-x) \le b(n-x)$ and $bn = 1 + s$. Thus, in spectral functional calculus,

$$0 \preceq Z - M \preceq qI, \qquad 0 \preceq H_R \preceq (1-p)I_R,$$

where $Z = z(S)$ and $h(S) = H_R \otimes I_A$. The latter belongs to the reference algebra because S does. Set $F = s(Z - M)$. Then

$$(\mathrm{id}_R \otimes \mathcal{N}_s)^*(M) = bS - sM = F + H_R \otimes I_A, \quad 0 \preceq F \preceq pI. \tag{8}$$

This is an effect-dependent construction; it is not a linear CPTP postprocessing map.

4.3 Passage to all Hermitian inputs and the supremum

For Hermitian X, the variational identity $\operatorname{Tr} X_+ = \max_{0\le M\le I} \operatorname{Tr}(MX)$ and (8) give

$$\operatorname{Tr}(\mathcal{N}_{s,R}X)_+ \le p \operatorname{Tr} X_+ + (1-p)\operatorname{Tr}(X_R)_+.$$

Subtract the unchanged reference positive trace to obtain the stronger, unrestricted statement

$$g(\mathcal{N}_{s,R}X) \le p_n(s) g(X) \quad \text{for every Hermitian } X \text{ and every finite } R. \tag{9}$$

Taking $X = \rho - \gamma\sigma$ proves the coefficient upper bound. At $s = 0$, the output is the maximally mixed replacer and the conditional gap is zero. At $s = 1/(n-1)$, $p_n(s) = 1$ and ordinary channel data processing supplies the endpoint. The actual CPTP map is constructed from positive multiples of the identity map and $d\operatorname{Tr}(\cdot)I - \mathrm{id}$, with coefficients

$$\frac{1-(n-1)s}{n}, \qquad \frac{1+s}{n},$$

respectively. No CP assumption is left unproved for this family.

5 Matching lower bound in every dimension

5.1 Balanced frame and actual normalized states

Let $n = d^2$. Choose phases ζ_i of modulus one with $\sum_i \zeta_i = 0$. Explicitly index $i = (a,b) \in (\mathbb{Z}/d\mathbb{Z})^2$ and take $\zeta_i = e^{2\pi i a/d}$. Put

$$e_i^+ = \frac{1}{\sqrt{2}}(1,\zeta_i), \qquad e_i^- = \frac{1}{\sqrt{2}}(1,-\zeta_i).$$

Each vector has norm one, $e_i^+ \perp e_i^-$, and both families have frame operator $(n/2)I_2$. If Z has columns e_i^+, then

$$P = I_n - \frac{2}{n}Z^\dagger Z$$

is an orthogonal projection, $\operatorname{Tr} P = n-2$, $P_{ii} = r := 1-2/n$, and $PZ_\pm^\dagger = 0$. In Lean the equivalent common phase $(1+i)/2$ replaces $1/\sqrt{2}$; it has squared modulus $1/2$ and avoids unnecessary square roots.

Use $S_0 = \mathbb{C}^n \oplus \mathbb{C}^2 \oplus \mathbb{C}^n$ as an auxiliary space. For $u > 0$, let

$$v_i = (P\delta_i, ue_i^+, 0), \qquad w_i = (0, ue_i^-, P\delta_i), \qquad X_i = v_i v_i^\dagger - w_i w_i^\dagger.$$

Then $v_i \perp w_i$, both have squared norm $r + u^2$, and

$$\sum_i X_i = T_0 := \operatorname{diag}(P, 0, -P), \quad \|X_i\|_1 = 2(r+u^2), \quad \|T_0\|_1 = 2(n-2). \tag{10}$$

For example, the central sums cancel because the two frame operators agree, and the cross sums vanish by $PZ_\pm^\dagger = 0$.

Now use the actual orthonormal Bell vectors on $T \otimes A$, $\dim T = \dim A = d$,

$$|\phi_{a,b}\rangle = d^{-1/2} \sum_{x\in\mathbb{Z}/d\mathbb{Z}} e^{2\pi i b x/d} |x\rangle_T |x+a\rangle_A, \qquad \Phi_i = |\phi_i\rangle\langle\phi_i|.$$

Character orthogonality gives $\Phi_i\Phi_j = \delta_{ij}\Phi_i$, $\sum_i \Phi_i = I$, $\operatorname{Tr}\Phi_i = 1$ and $\operatorname{Tr}_A \Phi_i = I_T/d$. Define states on $(S_0 \otimes T) \otimes A$ by

$$\rho_u = \frac{\sum_i v_i v_i^\dagger \otimes \Phi_i}{n(r+u^2)}, \qquad \sigma_u = \frac{\sum_i w_i w_i^\dagger \otimes \Phi_i}{n(r+u^2)}. \tag{11}$$

Positivity and normalization follow term by term. They are orthogonal mixed states. The full reference is $S_0 \otimes T$, not S_0 alone, and

$$(\rho_u - \sigma_u)_R = \frac{T_0 \otimes I_T/d}{n(r+u^2)}.$$

Orthogonal Bell blocks preserve the sum of trace norms. Equation (10) therefore gives

$$\Delta_1(\rho_u, \sigma_u) = \frac{u^2}{r+u^2} > 0. \tag{12}$$

The reference dimension at $\gamma = 1$ is $(2n+2)d = 2d(d^2+1)$. No optimality of that dimension, or pure-state reduction, is asserted.

5.2 The output blocks and a complete dual certificate

Set $b = (1+s)/n$ and

$$a = b - sr = \frac{1-(n-3)s}{n} > 0.$$

Since $(\mathrm{id}_T \otimes \mathcal{N}_s)(\Phi_i) = bI - s\Phi_i$, the unnormalized output is Bell-block diagonal with blocks

$$Y_i = bT_0 - sX_i.$$

For a direct spectral check, each Y_i has two opposite 2×2 blocks represented by

$$\begin{pmatrix} a & -su\sqrt{r} \\ -su\sqrt{r} & -su^2 \end{pmatrix}$$

and its negative, together with eigenvalues b and $-b$ of multiplicity $n-3$ each and four zero eigenvalues. Its trace norm is $2\sqrt{(a+su^2)^2 + 4s^2ru^2} + 2b(n-3)$. Because $a + b(n-3) = r$, the resulting exact conditional ratio is

$$\frac{\sqrt{(a+su^2)^2 + 4s^2ru^2} - a}{u^2} \longrightarrow s + \frac{2s^2r}{a} = p_n(s). \tag{13}$$

The formal proof does not assume this spectral block formula. The following explicit rational dual certificate proves the needed lower bound without it.

Let K_i have the four columns $(P\delta_i, 0, 0)$, $(0, e_i^+, 0)$, $(0, e_i^-, 0)$, $(0, 0, P\delta_i)$. Then

$$G := K_i^\dagger K_i = \mathrm{diag}(r, 1, 1, r), \qquad T_0 K_i = K_i J, \quad J = \mathrm{diag}(1, 0, 0, -1).$$

With $z = 1 + rt^2$ and $c = (1 - rt^2)/z$, define

$$D(t) = \begin{pmatrix} -2t^2/z & -2t/z & 0 & 0 \\ -2t/z & -c & 0 & 0 \\ 0 & 0 & c & 2t/z \\ 0 & 0 & 2t/z & 2t^2/z \end{pmatrix}, \qquad W_i(t) = T_0 + K_i D(t) K_i^\dagger.$$

Direct multiplication gives

$$JD + DJ + DGD = \mathrm{diag}(0, 1, 1, 0).$$

The two middle columns resolve I_2, so

$$W_i(t)^2 = \mathrm{diag}(P, I_2, P) \preceq I.$$

Thus $W_i(t)$ is a Hermitian contraction and $\mathrm{Tr}[W_i(t)Y_i] \le \|Y_i\|_1$. A four-dimensional cyclic-trace expansion gives

$$\tfrac{1}{2}\,\mathrm{Tr}[W_i(t)Y_i] = b(n-3) + ac + su^2c + \frac{4srtu}{1+rt^2}. \tag{14}$$

This identity includes the entire ambient trace; none of the complementary eigenspaces is discarded.

Take $k = s/a$ and $t = ku$. Subtract the full reference norm and use $a + b(n-3) = r$. The actual state ratio is bounded below by

$$L(u) = \frac{p_n(s) - srk^2u^2}{1 + rk^2u^2}. \tag{15}$$

Indeed, after cancellation of the common normalization and the positive denominator in (12), the remaining identity is

$$a(c-1) + su^2c + \frac{4srtu}{1+rt^2} = u^2L(u).$$

All denominators are positive. Along the explicit sequence $u_m = 1/(m+1)$, $L(u_m) \to p_n(s)$. Each u_m defines genuine admissible states in (1); hence their ratios are bounded by its supremum. Taking the real limit proves $p_n(s) \le \eta_T^{\Delta}(\mathcal{N}_s)$, including both physical endpoints. Combined with (9), this proves the negative branch of Theorem 2.

5.3 The standard positive branch and comparison values

For $0 \le \lambda \le 1$, $\mathcal{D}_\lambda$ is a mixture of the identity and the maximally mixed replacer. Triangle inequality gives $g(\mathcal{D}_{\lambda,R}X) \le \lambda g(X)$. At $\gamma = 1$, orthogonal basis states with trivial reference attain ratio λ; the flag argument then proves the positive branch for every $\gamma \ge 1$. For either sign, if $X_R = 0$ then $\mathcal{D}_{\lambda,R}X = \lambda X$; the same basis witness proves $\eta_T^c = |\lambda|$.

For the negative branch, the scalar replacer $P = [1-(n-1)s]I_d/d$ is feasible. Conversely, the Choi matrix is

$$J_{\mathcal{N}_s} = \frac{1+s}{d} I - s|\Omega\rangle\langle\Omega|.$$

Testing $J_{\mathcal{N}_s} - I \otimes P \succeq 0$ against Ω gives $\mathrm{Tr}\,P \le 1-(n-1)s$. This proves the claimed α_+, including the endpoints. These comparison values are supporting calculations, not independently claimed discoveries.

6 Proof of the full Pauli formula

6.1 Symmetries and a convex upper bound

Input and output unitary conjugation preserve the full admissible ratio sets. Pauli translations and single-qubit Clifford rotations realize every needed relabeling of the four weights. It therefore suffices to put the two smallest weights in positions $0, 3$, with $w_0 = a$, $w_3 = b$ and $w_1, w_2 \ge b$.

The coefficient is convex in the channel. For a mixture $\mathcal{N} = \sum_j t_j \mathcal{N}_j$, joint convexity of the trace norm, followed by subtraction of the same reference norm, gives

$$g(\mathcal{N}_R X) \le \sum_j t_j\, g(\mathcal{N}_{j,R}X).$$

For the trace coefficient the corresponding uniform bound follows by dividing by the positive input gap and taking the supremum. Equivalently one may apply the full Hermitian upper bounds directly.

Put $m = a + 3b$. If $m > 0$, the Pauli core with normalized weights $(a, b, b, b)/m$ is negative qubit depolarization with $s = (b-a)/m \in [0, 1/3]$. The remaining channel mass consists of unitary conjugations, with weights $w_1 - b$ and $w_2 - b$. The exact qubit value already proved yields

$$\eta_T^{\Delta}(\mathcal{P}_w) \le m\frac{s(1+3s)}{1-s} + (1-m) = 1 - \frac{8ab}{a+b}.$$

If $m = 0$, then $a = b = 0$ and the general data-processing upper is one. This handles the entire boundary.

6.2 An explicit matching three-dimensional reference

Use the prior qubit–qutrit support vectors, credited to [5],

$$v_u = |0,0\rangle + u|1,1\rangle, \qquad w_u = u|1,0\rangle + |2,1\rangle, \qquad u > 0.$$

After dividing their outer products by $1 + u^2$, these are orthogonal pure density matrices on $\mathbb{C}^3 \otimes \mathbb{C}^2$. Their unnormalized difference X_u has trace norm $2(1 + u^2)$ and reference difference $\mathrm{diag}(1, 0, -1)$ with norm two. Their conditional input gap is $u^2/(1 + u^2)$.

Write $A = a + b$, $B = 1 - A$, $C = a - b$, and define

$$c(t) = \frac{1-t^2}{1+t^2}, \qquad d(t) = \frac{2t}{1+t^2}.$$

On the ordered basis $00, 01, 10, 11, 20, 21$, use the Hermitian reflection

$$W(t) = \begin{pmatrix} c & 0 & 0 & -d & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d \\ -d & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & d & 0 & 0 & -c \end{pmatrix}.$$

Its square is I. Expanding the actual Pauli map, including both coherences, gives

$$\tfrac{1}{2}\,\mathrm{Tr}[W(t)\mathcal{P}_{w,R}(X_u)] = B + Ac(t) + (B - A)u^2c(t) - 2Cu\,d(t).$$

The other Pauli coherence cancels from this pairing; it is not discarded from the channel. The trace-norm dual inequality and reference subtraction therefore bound the raw output gap below by

$$A[c(t) - 1] + (B - A)u^2c(t) - 2Cu\,d(t).$$

For $A > 0$ choose $k = -C/A$, $t = ku$. Dividing by the raw input gap u^2 gives the rational lower

$$\frac{P + (A - B)k^2u^2}{1 + k^2u^2}, \qquad P = B - A + \frac{2C^2}{A} = 1 - \frac{8ab}{a+b}.$$

Letting $u = 1/(m + 1) \to 0$ proves the matching supremum lower. If $A = 0$, use $u = 1, t = 0$ in the same actual dual pairing to obtain ratio one directly. This proves (3) without excluding zero weights or tied minima.

For the supporting Doeblin value, the scalar replacer $2aI_2$ is feasible because the unnormalized Pauli Choi eigenvalues are $2w_i$. Conversely, a Bell eigenvector for the least eigenvalue bounds $\mathrm{Tr}\,P \le 4a$. Hence $\alpha_+ = 4a$. The Doeblin upper is exact precisely when $a = 0$ or $a = b$; otherwise its excess is $4a(b - a)/(a + b) > 0$.

7 Sharp operational consequences and their limits

The following corollaries have complete mathematical proofs here. The formal archive checks the channel coefficients and comparison values; it does not introduce separate Lean definitions for erasure-channel degradation or multi-hypothesis guessing probability.

7.1 Binary comparison versus physical simulation

Let $\mathcal{E}_q$ be the quantum erasure channel with survival probability $0 \le q \le 1$. For Hermitian X and arbitrary reference,

$$\|\mathcal{E}_{q,R}(X)\|_1 = q\,\|X\|_1 + (1-q)\,\|X_R\|_1\,.$$

Consequently $\|\mathcal{N}_{s,R}(X)\|_1 \le \|\mathcal{E}_{q,R}(X)\|_1$ for all such X, R exactly when $q \ge p_n(s)$. Sufficiency follows from (9), and necessity follows from the normalized witnesses above. This is an exact binary statistical comparison including arbitrary priors and quantum references.

A CPTP degrading map $\mathcal{D}$ with $\mathcal{N}_s = \mathcal{D} \circ \mathcal{E}_q$ exists exactly when $q \ge (n-1)s$. Indeed, such a map restricts on the erasure flag to a density matrix τ and supplies a positive replacer component $(1-q)\,\mathrm{Tr}(\cdot)\tau$, forcing $1 - q \le \alpha_+$. Conversely the explicit optimal scalar replacer of trace α_+ supplies the decomposition at equality, and any larger survival probability follows by extra erasure before that map. Thus

$$q_{\text{physical}} - q_{\text{binary}} = \frac{(n-2)s[1-(n-1)s]}{1-(n-3)s} > 0$$

in the strict physical interior. The effect split (8) is nonlinear, which is consistent with the absence of a physical degrading map in this interval. General quantum comparison theory characterizes degradation by all reference-assisted ensemble-guessing tasks [7, Theorem 1, zero-error case]; no first-ever binary-versus-physical separation claim is made.

For fixed $0 < \theta < 1$ and $s = \theta/(n-1)$,

$$q_{\text{physical}} = \theta, \qquad q_{\text{binary}} = \frac{\theta(1+\theta)}{(1-\theta)(n-1)+2\theta}.$$

At $\theta = 1/2$, the thresholds are $q_{\text{physical}} = 1/2$ and $q_{\text{binary}} = 3/[2(d^2+1)]$, so their ratio is exactly $(d^2+1)/3$. The second threshold is $O(d^{-2})$. This comparison uses dimension-dependent negative noise, as required by complete positivity, and does not imply a fixed-noise asymptotic channel advantage.

d	s	complete trace	conditional/binary	physical simulation
2	1/5	1/5	2/5	3/5
3	1/16	1/16	3/20	1/2
10	1/198	1/198	3/202	1/2

These are substitutions into proved formulas, not numerical evidence for them.

7.2 An explicit multi-hypothesis witness

The n equiprobable Bell-complement inputs

$$\rho_i = \frac{I_n - \Phi_i}{n-1}$$

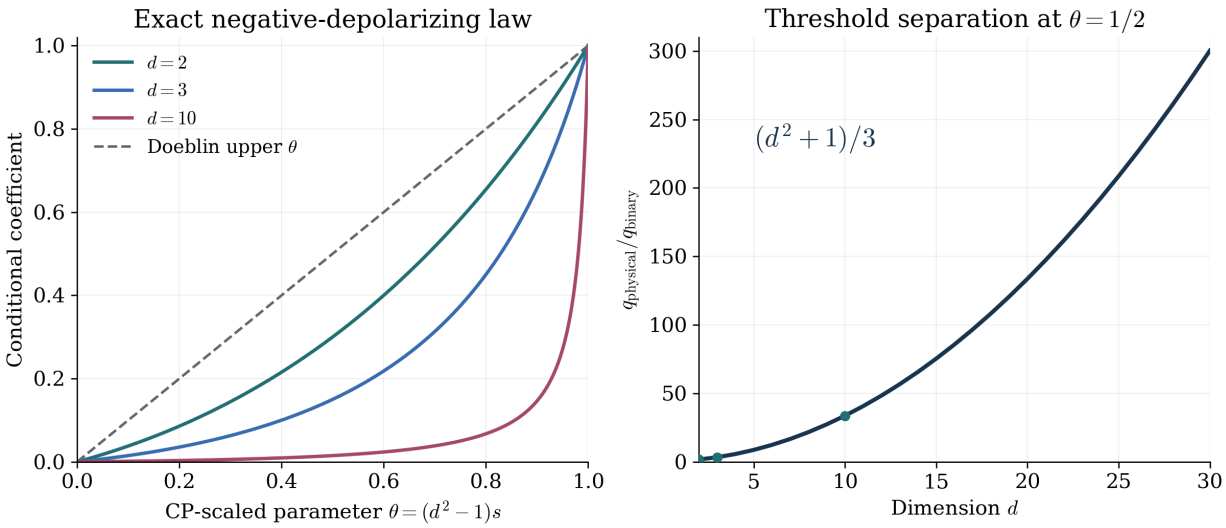


Figure 1: Evaluations of the exact formulas. Left: conditional contraction across the physical negative branch, with $\theta = (d^2-1)s$. Right: the ratio of the physical erasure-simulation threshold to the all-reference binary threshold at $\theta = 1/2$. The plotted values are consequences of the proofs, not numerical evidence for them.

have the same reference marginal $\rho_{i,T} = I_T/d$. Under $\mathcal{N}_s$ their outputs are $[(1-b)I + s\Phi_i]/(n-1)$. The Bell projective measurement is optimal: all outputs commute, and for each Bell outcome its own label has the largest posterior weight. Hence

$$P_{\text{guess}}(\mathcal{N}_s) = \frac{1+s}{n}.$$

For the erasure channel, successful transmission gives optimal guessing probability $1/(n-1)$; the erasure branch carries no label information and gives $1/n$. Thus

$$P_{\text{guess}}(\mathcal{E}_q) = \frac{n-1+q}{n(n-1)}.$$

The first exceeds the second exactly when $q < (n-1)s$, detecting the entire failure-of-degradation interval, including the region where every binary reference-assisted test is dominated. This is a concrete consequence of the exact thresholds, not an asserted new general comparison theorem.

8 Related work and scope

The closest inspected primary results are separated by their actual optimization domains. The conditional framework and the Doeblin upper bound are prior [1, 2]. Hiai and Ruskai [4] give ordinary unital-qubit trace contraction through the Bloch operator norm; their metric contractions concern another functional and do not subtract an arbitrary unequal reference marginal. Choi and Li [3] supply the standard unital-qubit normal form.

Eriksson's unital-qubit manuscript [5] determines equal-reference SLD, quadratic chi-square and triangular Petz quantities through a four-weight harmonic expression. It already contains the qubit–qutrit support geometry used above and dimension-dependent depolarizing Fisher/Petz laws. Its functionals and matching bounds differ from the trace-gap problem here. The common-support manuscript [6] is also prior; its simplex/support machinery and qualitative strictness results are credited prior ingredients.

The literature comparison is incomplete and does not establish first priority. The exact classification for the full Pauli family and the dimension-uniform depolarizing theorem are stated

independently of that historical question. No claim is made to resolve the general pure-state restriction question, relative-entropy contraction, tensorization, channel capacities, or finite-reference computability for arbitrary channels.

9 Formal verification and reproducibility

Verification passed on 29 October 2026. All 56 included local proof modules were rebuilt from source with zero compiler errors. The final audit checked 42 essential exported declarations; every reported axiom belonged to the standard foundation set specified below. The actual full signed depolarizing-channel constructor and the final concrete Fourier/Bell coefficient theorem are included in this successful build.

The formalization represents a quantum channel as a complex-linear map with proved complete positivity and trace preservation. Density matrices have PSD and trace-one proofs. The trace norm is the sum of absolute values of actual Hermitian eigenvalues. Partial traces are explicit matrix sums, and the coefficient is a real supremum of actual positive-denominator ratios over all finite references. The code proves the channels, witnesses, normalization, norm bounds, ratio membership, limiting passage and matching upper bound; a scalar identity alone is not treated as verification of a contraction theorem.

The pinned environment is Lean 4.19.0 with Mathlib commit

`c44e0c8ee63ca166450922a373c7409c5d26b00b`

The verifier deletes local proof object files, rebuilds every included local source with `--trust=0-j1-DwarningAsError=true`, checks the pinned dependency checkout and source hashes, and inspects the transitive axiom dependencies of the essential exported statements. Its allowed standard foundations are propositional extensionality, classical choice and quotient soundness. No added mathematical axioms or unfinished proof holes are permitted.

The public-facing theorem files are the Pauli exact-law module, the final depolarizing exact-law module, the positive-branch module and the comparison modules. Intermediate Bell, frame, scalar and effect modules are included as source, with their actual dependencies. The supplied verification record and theorem audit identify the checked source hashes, module count and exported statements.

To reproduce with the pinned project, install its Lean toolchain, obtain the matching Mathlib dependency and cache, then run

```
lake exe cache get
python3 verify.py
```

An existing clean pinned Mathlib checkout can be used with

```
python3 verify.py --lean /path/to/lean \
  --mathlib /path/to/pinned/mathlib4
```

The source package includes no local compiled proof objects. The formal verification certifies the stated mathematics in its definitions; it does not establish novelty, significance, peer review, or correspondence of every additional prose corollary to a separate formal theorem.

10 Research process and AI disclosure

This work is fully autonomous agentic research. Autonomous AI research agents produced the research, proofs, formalization and earlier versions of this manuscript. Opus 5.5 by Anthropic used

with Claude Code revised the presentation of the manuscript on 1 November 2026: it rewrote the front matter and abstract, added the note on earlier public work and consolidated this disclosure. The revision does not change the mathematics. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed, and no expert human validation is claimed. The formal scope is stated in Section 9, and the limits of the literature comparison in Section 8.

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