Diagonal concavity does not imply concavity for sums of elementary symmetric polynomials

Dev D. Goyal

26 October 2026; revised 1 November 2026

Abstract

An explicit family disproves Lachaume's diagonal concavity conjecture [2] (Conjecture 1), which asserts that the pth root of a nonnegative combination of elementary symmetric polynomials is concave on the positive orthant if and only if it is concave on the positive diagonal. The counterexample has four variables and strictly positive coefficients. For the family $f_M = 1 + M\sigma_1 + \sigma_2 + M\sigma_3 + \sigma_4$, where $M > 0$, the fourth root is concave on the positive diagonal exactly when $M \geq 1$. It is concave on the full positive orthant exactly when $M = 1$. Two polynomial identities prove the failure of concavity. Concavity at $M = 1$ follows from the arithmetic–geometric mean inequality. A Lean 4 formalization proves both classifications and a rational midpoint certificate with Mathlib's concavity predicate and elementary symmetric functions.

1 The diagonal concavity conjecture

For $x = (x_1, x_2, x_3, x_4)$, define the elementary symmetric polynomials by

$$\sigma_j(x) = \sum_{1 \leq i_1 < \cdots < i_j \leq 4} x_{i_1} \cdots x_{i_j}, \qquad \sigma_0(x) = 1.$$

Write $\Gamma = (0, \infty)^4$. A function H is concave on a convex set if

$$H(\theta x + (1-\theta)y) \geq \theta H(x) + (1-\theta)H(y) \quad (0 \leq \theta \leq 1)$$

for all x, y in that set. Lachaume's Conjecture 1 [2] asserts an equivalence between concavity on the positive orthant and concavity on its positive diagonal for

$$\left(\sum_{j=0}^{p} a_j \sigma_j \right)^{1/p}, \qquad a_j \geq 0.$$

Here the positive diagonal consists of the vectors $(t, \ldots, t)$ with $t > 0$.

Theorem 1. *For $M > 0$, put*

$$f_M = 1 + M\sigma_1 + \sigma_2 + M\sigma_3 + \sigma_4, \qquad H_M = f_M^{1/4}.$$

Then

$$\begin{aligned} t \longmapsto H_M(t,t,t,t) \text{ is concave on } (0,\infty) &\iff M \geq 1, \\ H_M \text{ is concave on } \Gamma &\iff M = 1. \end{aligned}$$

Thus each $M > 1$ disproves Conjecture 1. In particular, $(a_0, a_1, a_2, a_3, a_4) = (1, 2, 1, 2, 1)$ is a counterexample with all coefficients positive.

Each term in f_M is nonnegative on Γ, and the constant term is one. Thus the fourth root is defined and smooth throughout the domain. The polynomial has degree four, every coefficient is positive, and the counterexample lies in the interior of the domain.

2 Two curvature identities

For a positive twice differentiable function q,

$$(q^{1/4})'' = \frac{4qq'' - 3(q')^2}{16q^{7/4}}. \tag{1}$$

A twice differentiable function on an open interval is concave exactly when its second derivative is nonpositive. This follows from the mean value theorem and the monotonicity of its first derivative.

The diagonal polynomial is

$$P_M(t) = f_M(t,t,t,t) = 1 + 4Mt + 6t^2 + 4Mt^3 + t^4.$$

Expansion gives

$$3(P_M'(t))^2 - 4P_M(t)P_M''(t) = 48(M^2-1)(t^2-1)^2. \tag{2}$$

If $M \geq 1$, equations (1) and (2) give concavity on the diagonal. If $0 < M < 1$, the second derivative is positive for every $t > 0$ except $t = 1$. This proves the first classification. It also excludes concavity on the full positive orthant when $0 < M < 1$. A concave function remains concave when restricted to an affine line.

To treat $M > 1$, let

$$x_* = (2, 2, 1/2, 1/2), \qquad v = (2, 2, 1, 1), \qquad Q_M(s) = f_M(x_* + sv).$$

This line lies in Γ for $s > -1/2$. Direct expansion gives

$$\begin{aligned} Q_M(s) = 4s^4 + (12M+12)s^3 + (28M+26)s^2 \\ + 27(M+1)s + (41+40M)/4. \end{aligned}$$

Thus

$$Q_M(0) = \frac{41+40M}{4}, \qquad Q_M'(0) = 27(M+1), \qquad Q_M''(0) = 52 + 56M.$$

It follows that

$$4Q_M(0)Q_M''(0) - 3(Q_M'(0))^2 = (M-1)(53M+55) > 0. \tag{3}$$

By (1), the second derivative of H_M along this line is positive at the interior point. Hence H_M is not concave when $M > 1$.

3 Concavity when the parameter is one

It remains to prove concavity when $M = 1$. The subset expansion gives

$$f_1(x) = \prod_{i=1}^{4}(1 + x_i).$$

To prove concavity, use the following geometric-mean inequality. For $u \in (0,\infty)^4$, put $G(u) = (u_1u_2u_3u_4)^{1/4}$. The arithmetic–geometric mean inequality implies, for positive u, z,

$$G(u) \leq \frac{G(z)}{4}\sum_{i=1}^{4}\frac{u_i}{z_i}. \tag{4}$$

For positive u, w, take $z = \theta u + (1-\theta)w$, where $0 \leq \theta \leq 1$. Apply (4) to u and w. Multiply each inequality by its corresponding weight and add. This gives

$$\theta G(u) + (1-\theta)G(w) \leq \frac{G(z)}{4}\sum_{i=1}^{4}\frac{\theta u_i + (1-\theta)w_i}{z_i} = G(z).$$

Thus G is concave. Composition with the affine map $x \mapsto (1+x_1, \ldots, 1+x_4)$ proves concavity of H_1. Together with the preceding section, this proves Theorem 1.

4 A finite midpoint witness

A separate rational certificate proves the failure at $M = 2$ directly from the concavity inequality. Let

$$x_- = (3/2, 3/2, 1/4, 1/4), \qquad x_+ = (5/2, 5/2, 3/4, 3/4).$$

Their midpoint is x_*. Exact evaluation gives

$$f_2(x_-) = 933/64, \quad f_2(x_*) = 121/4, \quad f_2(x_+) = 3597/64.$$

The rational comparisons

$$(977/500)^4 < 933/64, \qquad (1369/500)^4 < 3597/64, \qquad 121/4 < (4691/2000)^4$$

give

$$\frac{H_2(x_-) + H_2(x_+)}{2} > \frac{1173}{500} > \frac{4691}{2000} > H_2(x_*).$$

This is the strict reverse of the required midpoint inequality.

5 Formal verification and scope

The Lean namespace `MathAndTheory.DiagonalConcavity` proves the polynomial identities, derivative formulas, and concavity statements. The declaration `complete_classification` proves Theorem 1. The definition `elementary` uses Mathlib's `Multiset.esymm`. The theorem `family_eq_elementary` proves that this definition equals the coordinate formulas. The declaration `elementary_symmetric_counterexample` states the counterexample with these elementary symmetric functions. The declaration `midpoint_strict_violation` proves the midpoint certificate.

The result disproves Conjecture 1. It does not classify arbitrary coefficient vectors.

Lachaume's separate sufficient condition requires a real-rooted diagonal polynomial. The counterexample does not contradict that condition.

The Walsh-convolution note [1] concerns Conjectures 4 and 5. It makes no claim for Conjectures 1 and 3. No claim of first priority is made.

6 Research process and AI disclosure

This work is fully autonomous agentic research. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT. GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities; this check is not expert validation or peer review. Dev D. Goyal is responsible for publication. This preprint has not been externally peer reviewed.

Opus 5.5 by Anthropic used with Claude Code made the editorial revision of 1 November 2026 to the front matter, the abstract and this section; it did not change the mathematics. Section 5 states the formal scope and the limits of the result.

References

[1] Carptopus. A walsh-convolution proof of two conjectures of lachaume, September 2026. URL https://github.com/Carptopus/MathExplore-Notes/blob/6e96176abbbb884906b2235b418d2315908bb5bd/research/lachaume-walsh-convolution/README.md. Public version dated 4 September 2026. The source states that it addresses Conjectures 4 and 5, not Conjectures 1 and 3.

[2] Xavier Lachaume. On the concavity of a sum of elementary symmetric polynomials, 2017. URL https://arxiv.org/abs/1712.10327v1. Conjecture 1; submitted 12 December 2017.