Counterexamples to a late-iterate ordering conjecture for generalized Hofstadter functions

Dev D. Goyal

26 October 2026; revised 1 November 2026

Abstract

Letouzey, Li, and Steiner conjecture a pointwise order between iterates of successive generalized Hofstadter functions [3]. This paper disproves that order. A finite witness gives $F_4^9(132) = 8 > 7 = F_5^{10}(132)$. An inverse recurrence gives a counterexample for every parameter $k \geq 4$, at the first iteration index $j = 2k + 1$ in the conjectured range. The proof uses an exact difference of $k - 3$ between two inverse iterates. The Lean formalization constructs the functions on all natural numbers from their defining recurrence and proves both the finite witness and the full parameter family.

1 The conjecture and a finite counterexample

For $k \geq 1$, define

$$F_k(0) = 0, \qquad F_k(n+1) = n + 1 - F_k^k(n). \tag{1}$$

Here f^r means r-fold composition, with f^0 the identity. These are the functions in Section 1.1 of [3]. The unnumbered conjecture immediately before Lemma 7.6 in arXiv version 2 asserts

$$\forall k \geq 1\ \forall j > 2k\ \forall n \in \mathbb{N}, \qquad F_k^j(n) \leq F_{k+1}^{j+1}(n). \tag{C}$$

One counterexample is sufficient to disprove this universal statement. Theorem 1 below gives a finite counterexample, and Theorem 2 gives a counterexample for every $k \geq 4$ at the first iteration index $j = 2k+1$ allowed by (C). Lean proves both theorems from the defining recurrence (Section 3). Section 4 compares the results with earlier work.

Theorem 1 (Finite counterexample)**.** *The conjecture* (C) *is false. More precisely,*

$$\boxed{F_4^9(132) = 8 > 7 = F_5^{10}(132).} \tag{2}$$

The functions F_4^9 and F_5^{10} are incomparable in pointwise order, since also

$$F_4^9(36) = 2 < 3 = F_5^{10}(36). \tag{3}$$

Proof. The defining recurrence gives the following exact iteration chains:

$$132 \xrightarrow{F_4} 96 \to 70 \to 51 \to 37 \to 27 \to 20 \to 15 \to 11 \to 8,$$

$$132 \xrightarrow{F_5} 100 \to 75 \to 56 \to 42 \to 32 \to 25 \to 19 \to 14 \to 10 \to 7.$$

The Lean certificate checks every entry of the two recurrence tables through index 132, proves their agreement with the infinite functions in (1), and deduces (2) and (3). Since $9 > 2 \cdot 4$, (2) contradicts (C). The opposite strict inequalities at 132 and 36 prove incomparability. Section 3 gives the exact formal statements. □

2 An infinite family

The defining recurrence gives a counterexample for every parameter in the range below. The proof uses the inverse relation established in the next lemma. For $k \geq 1$, set

$$L_k(n) = n + F_k^{k-1}(n). \tag{4}$$

Theorem 2 (Failure for every parameter $k \geq 4$)**.** *For each integer $k \geq 4$, let*

$$p = k + 3, \qquad B_k = L_{k+1}^{2k+2}(p).$$

Then

$$\boxed{F_k^{2k+1}(B_k) > k + 3 = F_{k+1}^{2k+2}(B_k).} \tag{5}$$

In particular the proposed pointwise ordering fails for every $k \geq 4$ at the first integer j strictly greater than $2k$.

2.1 The inverse relation

Equation (1) defines a unique function on all natural numbers with $0 \leq F_k(n) \leq n$. Indeed, after all values below $n + 1$ have been constructed, every argument in $F_k^k(n)$ remains at most n. The subtraction therefore gives a nonnegative value and preserves the bound.

Lemma 1. *For $k \geq 1$, consecutive values of F_k differ by either zero or one. Moreover*

$$F_k(L_k(n)) = n, \qquad F_k(L_k(n) + 1) = n + 1, \tag{6}$$

and, for every $r, n, m \in \mathbb{N}$,

$$F_k^r(n) \leq m \quad \Longleftrightarrow \quad n \leq L_k^r(m). \tag{7}$$

In particular $F_k^r(L_k^r(m)) = m$.

Proof. The first difference is $F_k(1) - F_k(0) = 1$. Suppose the zero-or-one step property has been proved up to input $n + 1$. Iterating F_k preserves two ordered inputs at distance at most one within that range. The outputs remain bounded by the inputs. Hence $F_k^k(n+1) - F_k^k(n) \in \{0, 1\}$. Subtracting the two instances of (1) proves the next step. This also proves monotonicity, both for F_k and for all its iterates.

By (4), $L_k(n + 1) - L_k(n) \in \{1, 2\}$. The first identity in (6) follows by induction on n. It holds at zero. If $u = L_k(n)$ and $F_k(u) = n$, then (1) gives

$$F_k(u + 1) = u + 1 - F_k^{k-1}(n) = n + 1.$$

Put $v = L_k(n + 1)$. If $v = u + 1$, the induction step is complete. If $v = u + 2$, applying (1) once more gives

$$F_k(v) = v - F_k^{k-1}(n + 1) = n + 1.$$

The displayed calculation of $F_k(u + 1)$ also gives the second identity in (6). Monotonicity and (6) now imply $F_k(n) \leq m$ exactly when $n \leq L_k(m)$. Iteration proves (7). Iterating the first identity in (6) proves the final claim. These arguments also apply at zero. □

2.2 Recurrences for the inverse iterates

For a fixed starting value p, write $a_t = L_k^t(p)$. The cancellation identity in (6) and definition (4) imply, for all $t \geq 0$,

$$a_{t+k} = a_{t+k-1} + a_t. \tag{8}$$

To see this directly, apply (4) to $L_k^{t+k-1}(p)$ and cancel $k-1$ pairs of F_k and L_k.

Now take $k \geq 4$, $p = k+3$, and define

$$q_t = p + \frac{t(t+5)}{2}. \tag{9}$$

The small values obtained from (1) are

$$F_k(n) = n-1 \quad (2 \leq n \leq k+2), \qquad F_k(k+3) = k+1. \tag{10}$$

To prove these identities, first note that $F_k(1) = 1$. For $2 \leq n \leq k+2$, the previous values decrease by one until reaching the fixed point 1, so $F_k^k(n-1) = 1$. At $n = k+3$, the same argument gives $F_k^k(k+2) = 2$. Induction therefore proves (10).

Using (10) and cancellation in (4), the initial segments of the two relevant inverse orbits are

$$\begin{aligned} a_t &= L_k^t(p) = q_t, \qquad 0 \leq t \leq k-1, \\ b_t &= L_{k+1}^t(p) = q_t, \quad 0 \leq t \leq k. \end{aligned} \tag{11}$$

To calculate the increments, let $1 \leq t \leq k-1$. Cancellation of $t-1$ pairs gives

$$a_t - a_{t-1} = F_k^{k-1}(L_k^{t-1}(p)) = F_k^{k-t}(p).$$

The first application sends $p = k+3$ to $k+1$ by (10). Each of the remaining $k-t-1$ applications subtracts one. All these inputs lie between 2 and $k+2$. Thus $a_t - a_{t-1} = k+1-(k-t-1) = t+2$. For the other orbit, $1 \leq t \leq k$ gives

$$b_t - b_{t-1} = F_{k+1}^{k}(L_{k+1}^{t-1}(p)) = F_{k+1}^{k-t+1}(p).$$

Now $p = k+3 = (k+1)+2$ lies in the initial range of F_{k+1}. Each of the $k-t+1$ applications subtracts one, so the increment is again $k+3-(k-t+1) = t+2$. Both orbits start at p. Summing the increments $3, 4, \ldots, t+2$ proves (9) and (11).

Let $S = \sum_{t=0}^{k-1} q_t$ and $A_k = a_{2k+1}$. Summing (8) through one block, then taking the next two steps, yields

$$A_k = 3q_{k-1} + S + 2q_0 + q_1. \tag{12}$$

In detail, $a_{2k-1} = q_{k-1} + S$, $a_k = q_{k-1} + q_0$, and $a_{k+1} = q_{k-1} + q_0 + q_1$. Adding the last two terms to the first gives (12). The same calculation with recurrence order $k+1$ gives

$$B_k = b_{2k+2} = 3q_k + S + q_0. \tag{13}$$

Consequently

$$\begin{aligned} B_k - A_k &= 3(q_k - q_{k-1}) - q_0 - q_1 \\ &= 3(k+2) - (k+3) - (k+6) \\ &= k-3 > 0. \end{aligned} \tag{14}$$

The sum S cancels, so its closed form is not needed.

Proof of Theorem 2. By (6) iterated, $F_{k+1}^{2k+2}(B_k) = p$. Equation (14) says $B_k > L_k^{2k+1}(p)$. The equivalence (7) therefore gives $F_k^{2k+1}(B_k) > p$, proving (5). □

3 Formal verification and reproducibility

Lean defines the functions on all natural numbers by well-founded recursion. The theorem `validTable_correct` proves that a table satisfying the recurrence agrees with the function throughout its index range. The Lean kernel checks the tables for $k = 4, 5$ by reduction of `decide`.

The namespace `MathAndTheory.Hofstadter` contains the theorems `pointwise_incomparable` and `lateIterateOrderConjecture_false`. The foundation module proves the step property, both inverse identities, and the iterate equivalence (7). The family module proves `inverse_iterate_gap` and `general_counterexample`. The latter requires only $k \geq 4$. Every inverse identity it uses is proved from the recurrence.

An independent arithmetic check computes the tables as follows. Begin with $a_0 = 0$. At step $m \geq 1$, set $u = m - 1$. Replace u by a_u exactly k times, then append $m - u$. All lookup indices are less than m because $a_i \leq i$. The calculation therefore uses only integers and terminates at each step. This finite check supports the witness. Lean also proves the counterexample for every parameter in Theorem 2.

4 Relation to earlier work

Conjecture (C) concerns more iterations than the proved base comparison $F_k \leq F_{k+1}$ or the comparison at $j = k + 1$. Letouzey's preprint [1], deposited on 26 October 2026, proves the earlier-index comparison and strict base ordering beyond an explicit bound. Its Section 7.2 concerns $F_k^{k+1} \geq F_{k+1}^{k+2}$.

The associated Rocq development [2] contains a witness to $F_k^j \not\geq F_{k+1}^{j+1}$ for $j \geq k + 2$. Its witness to failure of the opposite order applies when $j \leq 2k$. That range excludes $j = 2k + 1$, which is the index in Theorem 2. The inverse relations used here are due to the earlier research, although this formalization proves them again from the recurrence. No claim of first priority is made.

The infinite family varies k. Theorem 2 gives one input for each parameter. It does not assert infinitely many violating inputs for each fixed k or classify all pairs of iteration indices. The finite incomparability statement concerns the pair F_4^9, F_5^{10}.

5 Research process and AI disclosure

This work is fully autonomous agentic research. Autonomous AI research agents produced the research, proofs, computations, formalization and drafts. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT, and GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities. This check is not expert validation or peer review.

Opus 5.5 by Anthropic used with Claude Code revised this manuscript editorially on 1 November 2026: it revised the abstract and this disclosure, added a summary of the results to the first section and labelled the theorem references. It did not change the mathematics.

Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed. Section 3 states the formal scope, and Section 4 states the relation to earlier work and the limits of the result.

References

[1] Pierre Letouzey. Re-ordering the generalized Hofstadter functions G, H, and beyond, October 2026. URL https://inria.hal.science/hal-05839193v1. HAL preprint hal-05839193v1; submitted 26 October 2026.

[2] Pierre Letouzey. Hofstadter functions: Rocq formalization, 2026. URL https://github.com/letouzey/hofstadter_g/tree/de2bc29a0b3a7dc1f57baf1131353856ee74ddf7. Commit de2bc29a0b3a7dc1f57baf1131353856ee74ddf7; accessed 26 October 2026.

[3] Pierre Letouzey, Shuo Li, and Wolfgang Steiner. Pointwise order of generalized Hofstadter functions G, H, and beyond. *Journal of Integer Sequences*, 29(3):Article 26.3.3, 2026. URL https://cs.uwaterloo.ca/journals/JIS/VOL29/Letouzey/letouzey3.pdf. arXiv:2410.00529v2, dated 22 May 2026.