An unbounded obstruction to projection AM–GM in Loewner order

A rational counterexample and an unbounded obstruction

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29 October 2026; revised 1 November 2026

Abstract

Can the symmetrized product of orthogonal projections be bounded in Loewner order by the corresponding power of their arithmetic mean? This comparison is motivated by the analysis of random-permutation coordinate descent by Sun, Luo and Ye [1]. A counterexample is given consisting of three rational rank-one projections in dimension three. More strongly, for every real constant C, three such projections can be chosen so that their symmetrized product is not bounded above by C times the cube of their mean. The mean in these examples is positive definite. The proofs are elementary exact matrix computations. The accompanying Lean 4 development checks the actual matrix statements, including projection identities and failure of positive semidefiniteness, for the rational example and for every real multiplier; the rank-one property, the positive-definite mean and the rational choices have conventional proofs. This note concerns the projection Loewner-order conjecture, not a disproof of an optimal coordinate-descent convergence rate. The same three projections were used earlier in a public refutation, and no priority is claimed for the three-projection counterexample. The relation to prior work is stated in Section 4.

1 The original problem

For real orthogonal projections $P_1, \dots, P_n \in \mathbb{R}^{N\times N}$, write

$$\mathcal{M}(P_1,\dots,P_n) = \frac{1}{n}\sum_{i=1}^{n} P_i, \qquad \mathcal{G}(P_1,\dots,P_n) = \frac{1}{n!}\sum_{\sigma\in S_n} P_{\sigma(n)}\cdots P_{\sigma(1)}.$$

Both matrices are symmetric: transpose reverses a permutation product, and reversal permutes the terms in the sum. The Loewner order $X \preceq Y$ means that $Y - X$ is positive semidefinite.

The projection arithmetic–geometric mean question is motivated by the work of Sun–Luo–Ye [1]. The comparison studied here is

$$\mathcal{G}(P_1,\dots,P_n) \preceq \mathcal{M}(P_1,\dots,P_n)^n \tag{1}$$

for arbitrary real orthogonal projections. The theorems below concern this explicit unrestricted formulation. The additional block structure in coordinate-descent applications is discussed in Section 4. The weaker comparison $\mathcal{G} \preceq \mathcal{M}$ is consistent with the examples. Section 5 states which parts of the two theorems are checked in Lean.

Earlier public work (note added 1 November 2026). On 31 October 2026 a check of public problem records found that the Omega Institute's public repository records a refutation of Conjecture 3.1 of Sun, Luo and Ye (issue #15647), with a claimed Lean proof committed

on 12 October 2026, about 17 days before this result was first committed to the author's repository [6]. It uses the same rank-one projections onto $(-1,1,1)^T$, $(1,-1,1)^T$ and $(1,1,-1)^T$ and the same quadratic form value $-8/243$ as Theorem 1. A later addition to that record, first committed on 20 October 2026, gives a six-dimensional counterexample for the block-structured target of issue #17674, which keeps the nonsingular block structure that this note leaves aside. Theorem 2, that no real multiplier repairs the inequality even with a positive-definite mean, does not appear in that record. No originality or priority claim is made for the three-projection counterexample.

2 A complete rational counterexample

Let $w = (1,1,1)^T$, let $J = ww^T$, and let I denote the 3×3 identity matrix.

Theorem 1. *Set*

$$u_1 = (-1,1,1)^T, \qquad u_2 = (1,-1,1)^T, \qquad u_3 = (1,1,-1)^T, \qquad P_i = \frac{1}{3} u_i u_i^T.$$

These are rational rank-one orthogonal projections. With $A = \mathcal{M}(P_1,P_2,P_3)$ *and* $G = \mathcal{G}(P_1,P_2,P_3)$,

$$A = \frac{4I - J}{9}, \qquad G = \frac{J - 2I}{81}, \qquad A^3 - G = \frac{82I - 30J}{729}. \tag{2}$$

In particular,

$$w^T(A^3 - G)w = -\frac{8}{243} < 0,$$

so (1) *is false for* $n = N = 3$.

Proof. Each u_i has squared norm 3, hence $P_i^T = P_i$ and $P_i^2 = u_i(u_i^T u_i)u_i^T/9 = P_i$. Since $u_i \neq 0$, its rank is one. Direct summation gives

$$\sum_i u_i = w, \qquad \sum_i u_i u_i^T = 4I - J,$$

which proves the formula for A. If i,j,k are distinct, $u_i^T u_j = u_j^T u_k = -1$, so

$$P_i P_j P_k = \frac{1}{27} u_i u_k^T.$$

For each ordered pair of distinct endpoints (i,k) there is exactly one remaining middle index. Thus

$$G = \frac{1}{162} \sum_{i \neq k} u_i u_k^T = \frac{1}{162} \left(ww^T - \sum_i u_i u_i^T \right) = \frac{J - 2I}{81}.$$

Finally $J^2 = 3J$, so $(4I - J)^3 = 64I - 21J$. Subtracting G yields (2). Since $Jw = 3w$ and $w^T w = 3$, the displayed negative quadratic form follows. □

3 No finite multiplicative correction

Theorem 2. *For every* $C \in \mathbb{R}$ *there are three rank-one real orthogonal projections* $P_1, P_2, P_3 \in \mathbb{R}^{3\times 3}$ *such that*

$$\mathcal{G}(P_1,P_2,P_3) \not\preceq C\, \mathcal{M}(P_1,P_2,P_3)^3.$$

The projections may be chosen with positive-definite arithmetic mean. For every prescribed real C, *they may also be chosen to have rational entries.*

Proof. For a parameter $r > 0$, put $d = 1 + 2r^2$, $b = 1 - r^2$, and

$$v_i = (1 - r)w + 3re_i, \qquad P_i(r) = \frac{v_i v_i^T}{3d} \quad (1 \le i \le 3),$$

where e_i are the standard coordinate vectors. These have the identities

$$v_i^T v_i = 3d, \qquad v_i^T v_j = 3b \ (i \ne j), \qquad v_i^T w = 3, \qquad \sum_i v_i = 3w. \tag{3}$$

Thus each $P_i(r)$ is a nonzero symmetric idempotent of rank one.

Write $A_r = \mathcal{M}(P_1(r), P_2(r), P_3(r))$ and $G_r = \mathcal{G}(P_1(r), P_2(r), P_3(r))$. From (3),

$$A_r w = \frac{1}{d} w.$$

For distinct i, j, k, multiplication of the rank-one matrices gives

$$P_i(r)P_j(r)P_k(r)w = \frac{b^2}{d^3} v_i.$$

Every first index occurs twice among the six permutations. Averaging and using $\sum_i v_i = 3w$ therefore gives

$$G_r w = \frac{(r^2 - 1)^2}{d^3} w, \qquad A_r^3 w = \frac{1}{d^3} w. \tag{4}$$

Consequently,

$$w^T (C A_r^3 - G_r) w = \frac{3(C - (r^2 - 1)^2)}{(1 + 2r^2)^3}. \tag{5}$$

Choose $r = |C| + 2$. To see directly that $(r^2 - 1)^2 > C$, put $x = |C| \ge 0$. Then $r^2 - 1 = x^2 + 4x + 3 \ge x + 3$, and $(x+3)^2 > x \ge C$. The denominator of (5) is positive, so the quadratic form is negative. This proves the asserted Loewner failure.

The column matrix of the v_i is $V = 3rI + (1 - r)J$. It acts as multiplication by 3 on $\mathbb{R}w$ and by $3r$ on $w^\perp$. Since $A_r = VV^T/(9d)$, its eigenvalues are $1/d$ on $\mathbb{R}w$ and r^2/d on $w^\perp$, both positive. Finally choose an integer $r > |C| + 2$ instead; the same strict inequality holds and every $P_i(r)$ has rational entries. □

The obstruction is visible on a common eigenvector of A_r and G_r: the ratio of their relevant eigenvalues is

$$\frac{w^T G_r w}{w^T A_r^3 w} = (r^2 - 1)^2.$$

It therefore persists for positive-definite means and becomes arbitrarily large with the dimension and number of factors fixed. For $r \ge 1$, the condition number of A_r is r^2. This family does not rule out a comparison restricted to means with a prescribed condition-number bound.

Proposition 3 (Minimal number of factors)**.** *For two real symmetric matrices X, Y,*

$$\left(\frac{X + Y}{2}\right)^2 - \frac{XY + YX}{2} = \frac{1}{4}(X - Y)^2 \succeq 0.$$

Thus (1) holds for one or two projections in every dimension, and three is the smallest possible number of factors in a counterexample.

Proof. Expand the square. Because $X - Y$ is symmetric, $(X - Y)^2 = (X - Y)^T (X - Y)$ is positive semidefinite. □

This two-factor identity is standard, and appears already in Recht–Ré [2]; it is included only to locate the first possible failure. No claim of minimal ambient dimension is made.

4 Prior work and scope

The distinction between several related matrix comparisons is essential:

1. The projection comparison (1) uses orthogonal projections and Loewner order. Theorem 2 rules out every finite multiplicative repair for this formulation, with three factors.

2. The Recht–Ré norm conjecture allows general positive-semidefinite matrices and takes the spectral norm after averaging products. Lai–Lim [3] and De Sa [4] refuted that conjecture in 2020. Those results are not being claimed as new here. Lai–Lim also note that the unrestricted positive-semidefinite Loewner version fails for three factors, without providing a projection example in that passage.

3. De Sa's main norm counterexamples use matrices with eigenvalues $0, 1, 2$, which are not projections. His separate construction using projections concerns a different expression, with norms inside the expectation. Neither construction is automatically a prior counterexample to (1).

4. Kim–Lee–Yun [5], Appendix E.2, revisit the projection AM–GM route in the analysis of general quadratic Hessians. Their main results and conjectures involve convergence rates and second-moment operators, which are not settled by the present Loewner-order failure.

The closest results above concern different matrix hypotheses or different comparisons. The contribution here is the explicit upper-Loewner obstruction for projections and its unbounded ratio in fixed dimension. No unconditional claim of priority is made.

The motivating block-coordinate application imposes additional structure: $P_i = I - A_i A_i^T$, $A_i^T A_i = I$, and a square nonsingular concatenation $[A_1, \ldots, A_n]$. The three-dimensional rank-one family does not satisfy that additional block structure. The result establishes the failure of the unrestricted projection comparison (1) and the absence of a finite multiplicative correction. These results do not refute the scalar-coordinate specialization or establish a reversal of the algorithms' convergence comparison. The known weak inequality $\mathcal{G} \preceq \mathcal{M}$ is consistent with the examples.

5 Formal verification and reproducibility

The accompanying project uses Lean 4.19.0 and the following Mathlib commit:

`c44e0c8ee63ca166450922a373c7409c5d26b00b`.

The finite theorem is encoded using matrices over $\mathbb{R}$, symmetry and idempotence, the explicit average of all six products, and Mathlib's `Matrix.PosSemidef` predicate. The witness quadratic form is evaluated inside Lean's proof system. The parameterized theorem likewise connects the actual projection family to failure of positive semidefiniteness for every real constant C.

The principal declaration in the namespace `ProjectionCounterexample` is

`three_projection_counterexample`.

It asserts the existence of three real symmetric idempotents for which $A^3 - G$ is not positive semidefinite. The principal declaration in the namespace `ProjectionFamily` is

`no_finite_loewner_constant`.

It quantifies over every real C and asserts the same failure for $CA^3 - G$. Both use the displayed average of all six products, so every ordering occurs exactly once.

The rank-one property, positive-definite mean, rational choice, and two-factor statement have complete proofs in this paper. They are not included in the endpoints of these Lean

declarations. The formal sources and reproduction commands accompany this paper. The finite Python check uses exact rational arithmetic and covers Theorem 1; it does not establish the universal quantifier in Theorem 2.

Source availability and license

The mathematical source and formal verification files accompany this note. They are licensed under Apache-2.0. Lean and Mathlib retain their own licenses. Public reproducibility instructions and a declaration map are in the accompanying README.

6 Research process and AI disclosure

This work is fully autonomous agentic research. Artificial intelligence systems generated the research, proofs, formalization, and earlier versions of this manuscript. Opus 5.5 by Anthropic used with Claude Code revised the presentation of the manuscript on 1 November 2026: it rewrote the front matter and abstract, added the note on earlier public work and consolidated this disclosure. The revision does not change the mathematics. Dev D. Goyal is responsible for publication. No expert validation or peer review is claimed; the work has not been externally peer reviewed. The formal scope is stated in Section 5, and the scope relative to prior work and to the coordinate-descent application in Section 4.

References

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[2] B. Recht and C. Ré. *Toward a Noncommutative Arithmetic–Geometric Mean Inequality: Conjectures, Case-Studies, and Consequences*. COLT, PMLR 23, 11.1–11.24, 2012. https://proceedings.mlr.press/v23/recht12.html.

[3] Z. Lai and L.-H. Lim. *Recht–Ré Noncommutative Arithmetic–Geometric Mean Conjecture is False*. ICML, PMLR 119, 5608–5617, 2020. https://proceedings.mlr.press/v119/lai20a.html.

[4] C. De Sa. *Random Reshuffling is Not Always Better*. Advances in Neural Information Processing Systems 33, 2020. https://proceedings.neurips.cc/paper/2020/hash/42299f06ee419aa5d9d07798b56779e2-Abstract.html.

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[6] The Omega Institute. Problem dossier on the Sun–Luo–Ye projection-matrix AM–GM conjecture (issues #15647 and #17674), public repository `the-omega-institute/trureturing`; claimed Lean refutation committed 12 October 2026. https://github.com/the-omega-institute/trureturing/blob/b0a8d9a413a6aefee1b1aeb9fb084ddbe2798388/Problems/sun-luo-ye-2020-projection-matrix-amgm.md.