An entropy increase for a structurally disassortative DomiRank graph

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Abstract

A connected graph with 14 vertices and 32 edges disproves the interval monotonicity assertion in Conjecture 4.9 of Zhang and Zhao, arXiv:2610.00107v4 [2]. Its degree–log-degree edge covariance is strictly negative. All DomiRank scores are positive on the closed parameter interval $[251/1000, 63/250]$, which lies inside the stable spectral domain. The entropy increases between the two endpoints by more than $1/10000$ in natural-logarithm units. The proof uses exact rational formulas and finite logarithm bounds. The Lean formalization proves the claims about the graph, resolvent scores, interval positivity, covariance, and entropy comparison. A broader, unconditional entropy-decrease assertion in an earlier version of the source had already been refuted publicly by a graph that does not satisfy the covariance hypothesis of Conjecture 4.9.

1 Problem and definitions

Let G be a finite simple undirected graph with adjacency matrix A. Write $d = A\mathbf{1}$ for its degree vector. With dominance threshold one, the DomiRank score vector at parameter s is

$$\gamma(s) = (I + sA)^{-1}(sd). \tag{1}$$

For a connected graph with at least two vertices, the stable positive parameter domain is $0 < s < -1/\lambda_{\min}(A)$. Here $\lambda_{\min}(A)$ is the smallest eigenvalue of the real symmetric matrix A. On an interval where every score is positive, define

$$p_i(s) = \frac{\gamma_i(s)}{\sum_j \gamma_j(s)}, \qquad h(s) = -\sum_i p_i(s) \log p_i(s), \qquad H(s) = \frac{h(s)}{\log 2}.$$

Thus H is Shannon entropy in bits, and log denotes the natural logarithm. Truncating negative scores at zero has no effect on an interval of positive scores.

Choose an oriented edge (U, V) uniformly from the $2|E(G)|$ oriented edges. The structural covariance is

$$C = \operatorname{Cov}(d_U, \log d_V) = \frac{\sum_i (Ad)_i \log d_i}{\sum_i d_i} - \frac{\sum_i d_i^2}{(\sum_i d_i)^2} \sum_i d_i \log d_i. \tag{2}$$

Changing to base-two logarithms divides C by $\log 2$ and preserves its sign. The graph is *structurally disassortative* when $C \le 0$.

Zhang and Zhao relate this covariance to the initial entropy derivative. Their Conjecture 4.9 extends the proposed decrease to every interval of positive scores in the stable domain [2]. The interval can lie in any component of the positive-score set. The authors also give an example with positive covariance. That example does not satisfy the covariance hypothesis of Conjecture 4.9. The graph below has negative covariance and refutes this refined conjecture. Section 6 lists the Lean statements that check every claim of the main theorem.

Earlier public work (note added 1 November 2026). On 31 October 2026 a check of public problem records found that the Omega Institute's public repository records, in a dossier committed on 5 October 2026 (issue #13267), a refutation with a six-vertex graph of the broader unconditional assertion in Remark 4.6 of version 1 of Zhang and Zhao's paper, that entropy decreases for every connected nonregular graph [1]. That graph has positive degree–log-degree covariance, so it does not satisfy the hypothesis of Conjecture 4.9 of version 4, which the graph below refutes. The two results are consistent. No originality or priority claim is made for the refutation of the broader assertion.

2 The graph and the main result

Use vertices $0, 1, \dots, 13$. The edges are exactly the following unordered pairs:

- the edge $\{1, 2\}$ and the three edges in each of $\{3, 4, 5\}$ and $\{6, 7, 8\}$;
- every pair with one endpoint in $\{0, 1, 2\}$ and the other in $\{3, 4, 5, 6, 7, 8, 9\}$;
- the four pairs $\{9, j\}$ for $10 \le j \le 13$.

These disjoint lists contain $1 + 6 + 21 + 4 = 32$ edges. Every vertex is at distance at most two from vertex 9, so the graph is connected.

Theorem 1. *For this graph, put $s_- = 251/1000$ and $s_+ = 63/250$. Then $C < 0$. For every $s \in [s_-, s_+]$, the matrix $I + sA$ is positive definite and every coordinate of $\gamma(s)$ is positive. Moreover,*

$$h(s_-) + \frac{1}{10000} < h(s_+).$$

In particular, $H(s_-) < H(s_+)$, contrary to the stated interval monotonicity.

The proof first establishes the covariance sign and spectral stability. Exact score formulas then prove positivity on the interval. Rational bounds for the logarithms establish the entropy increase. The leaf probability is small and increases between the endpoints, but the proof accounts for the contributions of all vertices.

3 Covariance and spectral stability

The degree vector is

$$d = (7, 8, 8, 5, 5, 5, 5, 5, 5, 7, 1, 1, 1, 1)^T.$$

Substitution in (2) gives

$$C = \frac{159 \log 5 - 153 \log 7 - 120 \log 2}{512} < 0. \tag{3}$$

For the strict sign, use $\log 5 \le \log 7 < 3 \log 2$ and $\log 2 > 0$. The numerator is at most $6 \log 7 - 120 \log 2$, which is strictly negative.

Lemma 2. *The matrix $15I + 4A$ is positive definite. Consequently $I + sA$ is positive definite for $0 \le s < 4/15$.*

Proof. Partition the vertices into classes

$$\{0\}, \quad \{1, 2\}, \quad \{3, 4, 5, 6, 7, 8\}, \quad \{9\}, \quad \{10, 11, 12, 13\}.$$

The space of vectors constant on each class is invariant under A. In the basis of class indicators, its quadratic form for $15I + 4A$ is

$$K = \begin{pmatrix} 15 & 0 & 24 & 4 & 0 \\ 0 & 38 & 48 & 8 & 0 \\ 24 & 48 & 138 & 0 & 0 \\ 4 & 8 & 0 & 15 & 16 \\ 0 & 0 & 0 & 16 & 60 \end{pmatrix}.$$

Its leading principal determinants are $15, 570, 22212, 116796, 1321488$, all positive. Sylvester's criterion therefore proves positivity on this five-dimensional space.

Its orthogonal complement has the following invariant decomposition. The difference of the coordinate vectors on $\{1, 2\}$ has adjacency eigenvalue -1. The two sum-zero triangle spaces have eigenvalue -1 and total dimension four. The difference between the two triangle indicators has eigenvalue 2. The sum-zero leaf space has eigenvalue 0 and dimension three. These dimensions sum to nine. Thus the remaining eigenvalues of $15I + 4A$ are 11 with multiplicity five, 23 with multiplicity one, and 15 with multiplicity three. They are positive. Finally,

$$I + sA = \left(1 - \frac{15s}{4}\right) I + \frac{s}{4}(15I + 4A).$$

Both summands have nonnegative quadratic forms, and the first is strictly positive on nonzero vectors when $s < 4/15$. □

Since $s_+ < 4/15$, this proves stability throughout the required interval. For any eigenpair $Av = \lambda v$ with $v \neq 0$ and $\lambda < 0$, positive definiteness gives $1 + s\lambda > 0$, hence $s < -1/\lambda$. This applies in particular to the smallest eigenvalue. The Lean proof establishes the same positivity statement through a full rational factorization of $15I + 4A$.

4 Exact scores on the whole interval

Put $x = s - 1/4$. Then $1/1000 \leq x \leq 1/500$ on $[s_-, s_+]$. Define

$$\begin{aligned} D(x) &= \frac{3}{16} - \frac{283}{32}x - \frac{59}{4}x^2 + 52x^3 + 92x^4 + 24x^5, \\ N_0(x) &= \frac{3}{4} - \frac{247}{8}x - \frac{19}{2}x^2 + 62x^3 + 24x^4, \\ N_1(x) &= \frac{3}{4} - \frac{259}{8}x + \frac{13}{2}x^2 + 86x^3 + 24x^4, \\ N_2(x) &= \frac{1}{4} - \frac{123}{8}x + \frac{69}{2}x^2 + 86x^3 + 24x^4, \\ N_3(x) &= \frac{3}{4} - \frac{327}{8}x - \frac{99}{2}x^2 + 62x^3 + 24x^4, \\ N_4(x) &= \frac{5}{8}x + \frac{77}{2}x^2 + 86x^3 + 24x^4. \end{aligned}$$

Each polynomial is positive on this interval. For D, N_0, N_1, N_2, N_3, discard all positive terms except the constant. Bound each remaining negative term using $x \leq 1/500$ and $x^2 \leq 1/250000$. The resulting rational lower bounds are positive. Every displayed term of N_4 is nonnegative and its linear term is positive.

The adjacency action on class-constant vectors is the matrix

$$B = \begin{pmatrix} 0 & 0 & 6 & 1 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Direct polynomial multiplication gives

$$(I + sB)(N_0, N_1, N_2, N_3, N_4)^T = D(7, 8, 5, 7, 1)^T.$$

Let $c(i)$ be the class index of vertex i. Since $I + sA$ is invertible,

$$\gamma_i(s) = s\frac{N_{c(i)}(s - 1/4)}{D(s - 1/4)}. \tag{4}$$

Thus the rational formulas in (4) equal the resolvent scores in (1). They prove strict positivity at every real parameter in the interval.

5 A rational entropy certificate

The common factors s and D^{-1} cancel when the scores are normalized. Table 1 gives the resulting probability of each vertex in a class. The multiplicities are $(1, 2, 6, 1, 4)$.

Class	$p_i(s_-)$	$p_i(s_+)$
0	89889445253/534258269542	168020629/988469056
1	89703948253/534258269542	167303879/988469056
2	29332448253/534258269542	107123383/1976938112
3	88634445253/534258269542	81549377/494234528
4	82948253/534258269542	685883/1976938112

Table 1: Exact endpoint probabilities, one value per vertex in each class.

For $0 \le t < 1$, integration of the geometric series gives

$$2\sum_{j=0}^{N-1}\frac{t^{2j+1}}{2j+1} \le \log\frac{1+t}{1-t} \le 2\sum_{j=0}^{N-1}\frac{t^{2j+1}}{2j+1} + \frac{2t^{2N+1}}{1-t^2}. \tag{5}$$

The upper remainder bound follows by replacing each remaining denominator $2j + 1$ by 1 and summing the resulting geometric tail. Using $t = 1/3$ and $N = 12$ gives

$$\frac{69314718055}{10^{11}} \le \log 2 \le \frac{69314718057}{10^{11}}.$$

For each probability p, choose r so that $1 \le 2^r p < 2$. Set $t = (2^r p - 1)/(2^r p + 1)$ and use (5) with $N = 7$. Then subtract $r \log 2$. All operations are rational. Table 2 records the resulting enclosures. Multiplication by the positive probabilities and the class multiplicities gives

	$10^8 \log p_i(s_-)$		$10^8 \log p_i(s_+)$	
Class	lower	upper	lower	upper
0	−178229885	−178229884	−177207058	−177207057
1	−178436459	−178436458	−177634555	−177634554
2	−290218503	−290218501	−291532324	−291532323
3	−179635882	−179635881	−180180147	−180180146
4	−877041770	−877041769	−796635274	−796635273

Table 2: Rational bounds for the logarithms in Table 1.

$$h(s_-) < \frac{2158577}{10^6}, \qquad \frac{2158712}{10^6} < h(s_+).$$

The entropy difference therefore exceeds $135/10^6 > 1/10000$. This completes the proof of Theorem 1. Dividing by $\log 2 > 0$ proves the entropy increase in bits. Dividing by $\log 14 > 0$ also proves an increase for normalized entropy.

6 Formal verification and reproducibility

The Lean sources are in `lean/MathAndTheory/Mathematics/DomiRank/`. Their namespace is `MathAndTheory.DomiRank`. The main theorem is `fourteen_vertex_counterexample`. The following declarations connect the formulas to the graph and entropy definitions:

Mathematical statement	Lean declaration
Graph adjacency and degrees	`A_eq_adjMatrix`, `degree_eq_graph_degree`
Connectivity	`graph_connected`
Covariance sign	`degreeLogCovarianceBits_neg`
Stability on the interval	`resolvent_posDef_on_interval`
Source spectral condition	`spectral_bound_on_interval`
Inverse formula and positivity	`gamma_eq_candidate`, `gamma_pos_on_interval`
Normalization correspondence	`distribution_eq_theoretical`
Strict rational entropy gap	`entropyNats_gap`
Source-entropy comparison	`theoretical_entropy_strict_increase`

Lean checks the spectral condition without a numerical eigenvalue assumption. It also checks the finite rational inequalities that bound the logarithms. The formalization uses Lean 4.35.0 and Mathlib commit `56a005855ea17e0b0958f73ec801ae140b0a1679`. The repository verification record gives the build, axiom audit, and independent-kernel commands and results.

The independent Python program reconstructs the edges from the three rules above. It uses exact matrix inversion, rational root isolation for the whole interval, and an independent rational factorization of $15I + 4A$. It also recomputes the oriented-edge covariance and the finite logarithm certificate. An 80-decimal numerical check gives

$$h(s_+) - h(s_-) \approx 0.000136399223471066814.$$

The rational certificate proves the sign of this gap. The program, versions, commands, and output are in `papers/mathematics/domirank-entropy/verification/`.

The theorem concerns the specified mathematical centrality model. It does not assess empirical performance on networks or settle a conjecture restricted to the first component of the positive-score set. No claim of first priority is made.

7 Research process and AI disclosure

This work is fully autonomous agentic research. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT. GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities; this check is not expert validation or peer review. Opus 5.5 by Anthropic used with Claude Code revised the presentation of the paper on 1 November 2026: it updated the front matter and abstract, added the note on earlier public work and renamed this section. The revision does not change the mathematics. Dev D. Goyal is responsible for publication. This preprint has not been peer reviewed. The formal scope and the limits of the result are stated in Section 6.

References

[1] The Omega Institute. Problem dossier on DomiRank entropy monotonicity (issue #13267). Public repository `the-omega-institute/trureturing`, committed 5 October 2026. URL `https://github.com/the-omega-institute/trureturing/blob/3a5b8280cfdf4a7cf1f5002f6a503d10f741614d/Problems/zhang-zhao-2026-domirank-entropy-monotonicity.md`.

[2] Yingying Zhang and Chengye Zhao. Structural properties of DomiRank centrality: Symmetry axioms, an entropy bound, and spectral interpolation, 2026. URL https://arxiv.org/abs/2610.00107v4. Version 4, 15 October 2026; Conjecture 4.9.