A quartic entropy deficit for random rankings

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Abstract

Let P be any distribution on the rankings of an n-element set. If every pairwise comparison has probability at distance at least $\varepsilon > 0$ from $1/2$, then

$$\mathrm{H}(P) \le \left(1 - \frac{\varepsilon^4}{144}\right) \log(n!).$$

The bound holds for every $n \ge 0$, without a transitivity or independence assumption. The proof combines the squared energy of the mean comparison matrix, a normalized cut obtained from balanced splitting, and an exact entropy bound for binary interleavings. Factorial induction removes any large-size threshold. This gives an explicit quartic deficit in the general qualitative theorem of Acan, Devlin and Kahn [2], which proved an entropy conjecture of Leighton and Moitra [1]; Devlin proposed a deficit of the form $C\varepsilon^4$ [3]. A Lean 4 formalization proves the theorem for every n, using finite probability distributions and the actual comparison events of complete rankings.

1 Statement and relation to prior work

A ranking of a finite set V is a list containing every element exactly once, equivalently a bijection $V \to \{0, \dots, n-1\}$. Write $u \prec v$ when u occurs earlier. All logarithms are natural, and

$$\mathrm{H}(P) = -\sum_{\rho \in \mathfrak{S}_V} P(\rho) \log P(\rho), \qquad 0 \log 0 := 0.$$

Changing both entropy and logarithms to another fixed base greater than one leaves the coefficient in the theorem unchanged.

Theorem 1. *Let V be a finite set of cardinality n, and let P be a probability distribution on its rankings. Suppose $\varepsilon > 0$ and*

$$\left|\mathbb{P}_P(u \prec v) - \frac{1}{2}\right| \ge \varepsilon \quad \text{for every distinct } u, v \in V. \tag{1}$$

Then

$$\mathrm{H}(P) \le \left(1 - \frac{\varepsilon^4}{144}\right) \log(n!). \tag{2}$$

For $n \le 1$ the ranking is unique and both sides vanish. For $n \ge 2$, the hypothesis itself forces $\varepsilon \le 1/2$; an extra upper-bound assumption is therefore unnecessary.

Leighton and Moitra introduced the entropy question in their work on poset extensions and proved a quartic bound when the preferred comparisons are consistent with a total order [1]. Acan, Devlin, and Kahn proved the general qualitative theorem [2]. Their general coefficient, at the end of Section 3, has dependence $\exp[-\varepsilon^{-O(1)}]$; their transitive Theorem 2 gives $(1 - \varepsilon^2/8) n \log n$. Devlin's research statement explicitly proposes a sufficient deficit of the form $C\varepsilon^4$ in the general log-factorial theorem [3].

Thus the qualitative local-to-global theorem is established prior work. The contribution claimed here is the explicit general quartic bound and the argument producing it. The constant and exponent are not claimed to be optimal.

2 The mean comparison matrix and its energy

Define

$$B_{uv} = \mathbb{P}_P(u \prec v) - \mathbb{P}_P(v \prec u).$$

Then $B_{uu} = 0$, $B_{vu} = -B_{uv}$, and, for $u \neq v$, $B_{uv} = 2\mathbb{P}_P(u \prec v) - 1$. Put $b = 2\varepsilon$. The hypothesis gives $|B_{uv}| \geq b$ for distinct labels.

For an ordering $\tau = (v_1, \ldots, v_m)$ of some subset of V, define

$$F_B(\tau) = \sum_{i<j} B_{v_i v_j}.$$

Lemma 2. *For a random full ranking ρ with law P,*

$$\mathbb{E}_P F_B(\rho) = \frac{1}{2} \sum_{u,v \in V} B_{uv}^2.$$

Consequently, if $n \geq 2$ and all distinct entries have modulus at least b, there is a ranking τ with $F_B(\tau) \geq b^2 n^2/4$.

Proof. For each unordered pair $\{u, v\}$, its contribution to the score is B_{uv} if u precedes v, and $B_{vu} = -B_{uv}$ otherwise. Its expected contribution is therefore

$$B_{uv}\big(\mathbb{P}(u \prec v) - \mathbb{P}(v \prec u)\big) = B_{uv}^2.$$

Summing over unordered pairs proves the identity. There are $n(n-1)$ ordered pairs of distinct elements, so the expectation is at least $b^2 n(n-1)/2 \geq b^2 n^2/4$. At least one value of a finite random variable is no smaller than its expectation. □

3 A normalized cut

For disjoint subsets $L, R \subseteq V$, write $B(L, R) = \sum_{u \in L, v \in R} B_{uv}$. When $n \geq 2$, the following maximum is over a finite nonempty family:

$$A = \max_{\substack{L,R \neq \varnothing \\ L \cap R = \varnothing}} \frac{|B(L,R)|}{(|L| + |R|)^{3/2}}.$$

Lemma 3. *Every ordering τ of a subset of V, of length s, satisfies $F_B(\tau) \leq 6As^{3/2}$.*

Proof. Induct on s. The assertion is immediate for $s = 0, 1$. For $s \geq 2$, split the list into consecutive parts of lengths $a = \lfloor s/2 \rfloor$ and $r = \lceil s/2 \rceil$. Both lengths are positive and at most $2s/3$. The score is the sum of the two child scores and the cut sum between their label sets. The latter is at most $As^{3/2}$. Moreover,

$$a^{3/2} + r^{3/2} \leq (a + r)\sqrt{2s/3} \leq \frac{5}{6} s^{3/2},$$

where the last inequality follows by squaring $\sqrt{2/3} \leq 5/6$. The induction hypotheses give a total at most $6A(5/6)s^{3/2} + As^{3/2} = 6As^{3/2}$. □

Lemma 4. *Under the hypotheses of Lemma 2, there are disjoint nonempty $L, R \subseteq V$ such that, with $q = |L| + |R|$ and $D = B(L, R)$,*

$$\frac{D^2}{q^3} \geq \frac{b^4 n}{576}. \tag{3}$$

Proof. Apply Lemma 3 to the ordering from Lemma 2. It gives $b^2 n^2/4 \leq 6An^{3/2}$, hence $A \geq b^2\sqrt{n}/24$. A pair attaining the maximum has $D^2/q^3 = A^2$, which proves the claim. The argument does not prescribe the sign of D. □

4 Entropy of a binary interleaving

Let $\Omega_{q,r}$ be the binary words of length q containing exactly r symbols R, and $q-r$ symbols L. Define

$$\begin{aligned} C(y) &= \#\{L \text{ before } R\} - \#\{R \text{ before } L\} \\ &= 2\sum_{j=1}^{q} j\,\mathbf{1}_{\{y_j=R\}} - r(q+1). \end{aligned}$$

To verify the identity, the signed sum over all positions against an R in position j is $2j-q-1$. Removing the R/R comparisons leaves the same total, because those terms cancel in pairs. Equivalently, expand the signed count over all pairs of positions and use antisymmetry.

Lemma 5. *Suppose $\alpha, \beta \geq 0$, $s \geq 0$, $|u|, |v| \leq s$, and $\alpha u + \beta v = 0$. For every $t \in \mathbb{R}$,*

$$\alpha e^{tu} + \beta e^{tv} \leq (\alpha+\beta)e^{s^2t^2/2}.$$

Proof. If $s = 0$, both u and v vanish. Otherwise convexity of the exponential on $[-s, s]$ gives, for $|x| \leq s$,

$$e^{tx} \leq \frac{s+x}{2s}e^{ts} + \frac{s-x}{2s}e^{-ts} = \cosh(ts) + \frac{x}{s}\sinh(ts).$$

The weighted linear terms cancel. Finally $\cosh z \leq e^{z^2/2}$: compare their even power series term by term using $(2k)! = \prod_{j=1}^{k}(2j)(2j-1) \geq 2^k k!$. All terms are nonnegative. This proves the bound for every real t, including negative t. □

Lemma 6. *For every $q \geq 0$, $0 \leq r \leq q$, and $t \in \mathbb{R}$,*

$$Z_{q,r}(t) := \sum_{y \in \Omega_{q,r}} e^{tC(y)} \leq \binom{q}{r} \exp(q^3t^2/2). \tag{4}$$

Proof. Induct on q. The case $q = 0$ has one empty word. For $q > 0$, the extreme slices $r = 0, q$ each have one word, with $C = 0$, and satisfy the estimate. For $1 \leq r < q$, splitting by the first symbol gives the exact recurrence

$$Z_{q,r}(t) = e^{tr}Z_{q-1,r}(t) + e^{t(r-q)}Z_{q-1,r-1}(t).$$

Indeed, a first L precedes all r later R symbols, while a first R precedes all $q-r$ later L symbols and contributes $r-q$. Set $\alpha = \binom{q-1}{r}$ and $\beta = \binom{q-1}{r-1}$. The binomial identities

$$\alpha + \beta = \binom{q}{r}, \qquad \alpha r + \beta(r-q) = 0$$

allow Lemma 5 with $s = q$. The induction hypotheses imply

$$Z_{q,r}(t) \leq \binom{q}{r}\exp\Big(((q-1)^3 + q^2)t^2/2\Big) \leq \binom{q}{r}e^{q^3t^2/2}.$$

The last step uses $(q-1)^3 + q^2 \leq q^3$. Impossible slices have zero partition function and zero binomial coefficient; the formal statement also includes them. □

Lemma 7. *If $q > 0$ and Y has any distribution supported on $\Omega_{q,r}$, then, with $D = \mathbb{E}C(Y)$,*

$$\mathrm{H}(Y) \leq \log\binom{q}{r} - \frac{D^2}{2q^3}. \tag{5}$$

For $q = 0$, the entropy and the statistic are zero.

Proof. The finite Gibbs inequality has the following elementary proof. For any finite support set S, probability masses p_x, and real numbers f_x, let $Z = \sum_{x \in S} e^{f_x}$ and $Q_x = e^{f_x}/Z$. On the positive support of p, apply $\log z \le z - 1$ to $z = Q_x/p_x$ and multiply by p_x. The resulting sum is at most $\sum_x Q_x - \sum_x p_x = 0$. Rearranging yields

$$\mathrm{H}(p) \le \log \sum_{x \in S} e^{f_x} - \sum_x p_x f_x.$$

This argument permits zero probabilities and needs no positivity assumption outside the support. Apply it with $f_y = tC(y)$ and then Lemma 6:

$$\mathrm{H}(Y) \le \log \binom{q}{r} + q^3 t^2/2 - tD.$$

The admissible choice $t = D/q^3$, of either sign, gives the assertion. □

For a ranking of $L \cup R$, record only whether each successive element lies in L or R. Its word statistic is exactly the sum of signed comparisons over $L \times R$. Therefore its expected statistic is $D = B(L, R)$, regardless of dependencies among comparisons. Combining Lemmas 4 and 7 yields an interleaving entropy loss of at least $b^4 n/1152$.

5 Factorial induction and proof of the theorem

The signed-mean version is

$$\mathrm{H}(P) \le (1 - c) \log(n!), \qquad c = b^4/2304, \tag{6}$$

whenever every distinct entry of the mean comparison matrix has modulus at least $b > 0$. Proceed by strong induction on n. The cases $n = 0, 1$ have one ranking and zero entropy.

For $n \ge 2$, select L, R by Lemma 4. Let $Q = L \cup R$, $W = V \setminus Q$, and write

$$a = |L|, \quad r = |R|, \quad q = a + r, \quad w = |W|, \quad n = q + w.$$

A full ranking is uniquely determined by its rankings on L, R, W, its inner L/R interleaving word Y, and its outer Q/W interleaving word Z. To reconstruct it, merge the L and R lists according to Y, then merge the resulting list with the W list according to Z. Thus the encoding is injective. Entropy is unchanged by an injective encoding and is at most the sum of the entropies of its coordinates. For completeness, an injective map retains each positive mass and only adds zero masses outside its image, so its entropy is unchanged. For subadditivity, compare a joint distribution to the product of its coordinate marginals. That product is positive wherever the joint mass is positive. The same $\log z \le z - 1$ argument as in Lemma 7 gives joint entropy at most the cross-entropy against that product. Expanding the logarithm of the product yields the sum of marginal entropies. Repeated application gives the required finite-coordinate bound. The formalization proves these identities directly from finite sums.

The distributions on the three restricted lists are the unconditional pushforwards of P. For two retained labels, restriction changes neither their relative order nor its probability. Each restricted law therefore satisfies the same bias condition. Since $a, r > 0$, all three sizes a, r, w are strictly smaller than n, even if $Q = V$ or W is empty. The induction hypotheses apply to every child.

By Lemma 7 and the selected cut,

$$\mathrm{H}(Y) \le \log \binom{q}{r} - \frac{b^4 n}{1152}, \qquad \mathrm{H}(Z) \le \log \binom{n}{q}.$$

For the second inequality it suffices to drop the nonnegative deficit in the same lemma; alternatively, use the size of the outer word's support. Consequently, on setting

$$T = \log(a!) + \log(r!) + \log(w!), \qquad M = \log \binom{q}{r} + \log \binom{n}{q},$$

this gives

$$\mathrm{H}(P) \le (1-c)T + M - 2cn. \tag{7}$$

Exact factorial cancellation gives $T + M = \log(n!)$. Moreover,

$$M \le (q+n)\log 2 \le 2n,$$

since each binomial coefficient is at most the number of all binary words of the same length, $q \le n$, and $\log 2 \le 1$. As $c \ge 0$, the right-hand side of (7) is at most

$$(1-c)(T+M) + cM - 2cn \le (1-c)\log(n!).$$

This closes the induction. Finally $b = 2\varepsilon$ gives $b^4/2304 = \varepsilon^4/144$, proving Theorem 1. □

6 Formal model and reproduction

The Lean model represents a finite probability distribution by nonnegative real masses whose sum is one. Entropy is the finite sum of `Real.negMulLog`. A ranking is a list with no repeated labels that contains every label. Restriction filters this list. Pushforward probabilities sum the masses over each inverse image. Comparison probabilities are expectations of event indicators.

The main theorem is

`PermEntropy.FinDist.entropy_of_pairwise_bias_all`.

The theorem `entropy_of_pairwise_bias_equiv_all` gives the same bound for bijections from the label set to its positions. The event in its hypothesis is the literal comparison of two positions. Both statements include every finite size. The central entropy and cut bounds are proved lemmas, not assumptions of either theorem.

The package pins Lean 4.19.0 and the Mathlib revision in `lake-manifest.json`. Its README gives the build and axiom-audit commands. The audit permits only `propext`, `Classical.choice`, and `Quot.sound`.

7 Scope and limitations

The hypothesis concerns every unordered pair, with a common positive lower bound on absolute bias. The preferred directions may contain cycles. The argument places no independence condition on the distribution, on restricted rankings, or on interleaving words. Its recursion uses unconditional restrictions throughout.

The cut maximum is used existentially. This proof does not provide a polynomial-time algorithm for finding the cut from a succinctly represented distribution, nor an algorithm for realizing a prescribed comparison matrix. It also does not determine the best exponent or constant, classify extremizers, or prove a matching lower construction. The uniform distribution is excluded by positive pairwise bias when $n \ge 2$.

The balanced ordering decomposition has antecedents in tournament-ranking arguments acknowledged in the cited literature. Gibbs' inequality, entropy subadditivity, and the exponential estimate are standard tools. The quartic bound combines these tools with the mean comparison energy, normalized cuts, and exact factorial induction.

8 Research process and AI disclosure

This work is fully autonomous agentic research. The initial research and proofs were generated autonomously by GPT 6 with Pro Reasoning in ChatGPT. GPT 6 Astra in Codex autonomously produced the formalizations and corrections. The author checked the proofs to the best of their mathematical abilities; this check is not expert validation or peer review. Dev D. Goyal is responsible for publication. The work has not been externally peer reviewed.

Opus 5.5 by Anthropic used with Claude Code made the editorial revision of 1 November 2026 to the front matter, the abstract, this section and the references; it did not change the mathematics. Section 6 states the formal scope and Section 7 the remaining limits.

References

[1] T. Leighton and A. Moitra. *On Entropy and Extensions of Posets*. Manuscript, 12 September 2011. https://people.csail.mit.edu/moitra/docs/poset.pdf.

[2] H. Acan, P. Devlin, and J. Kahn. Proof of an entropy conjecture of Leighton and Moitra. *Journal of Combinatorial Theory, Series A* **161** (2019), 299–308. doi:10.1016/j.jcta.2018.08.004. arXiv:1701.04321v2, 10 March 2017.

[3] P. Devlin. *Research Statement: Unabridged*. Undated primary document, pp. 1 and 5. Accessed 27 October 2026. https://sites.math.rutgers.edu/~prd41/employers/Research_statement_unabridged.pdf.

[4] The Lean and Mathlib communities. Lean 4 and Mathlib source repositories. https://github.com/leanprover/lean4; https://github.com/leanprover-community/mathlib4. Exact revisions appear in `lean-toolchain` and `lake-manifest.json`.