A polygon enclosure fails for sums of normal matrices

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Abstract

This note defines a polygon enclosure using segments between permutation values for sums of normal matrices, motivated by Kovačec's proposed strengthening of the Marcus–de Oliveira conjecture [1]. An explicit pair of normal 3×3 matrices gives a determinant outside this region. For spectra $(0, 1, i)$ and $(-1 - i, 1, -1 + i)$, a real orthogonal change of basis gives the exact determinant $2/3 + 2i/9$. The boundary defined by arbitrarily long straight segments that avoid the permutation graph is a simple closed polygon. The determinant lies outside the closure of every bounded component of its complement, and a nonempty bounded component exists. The proof identifies the entire graph, its straight-segment exterior, and a Jordan parametrization. The same determinant is an explicit convex combination of permutation values, so the example respects the ordinary convex-hull inclusion. A companion Lean development verifies the matrix and topological statements. Identification with an external enclosure proposal and historical novelty remain unverified. Professor Kovačec found a counterexample to his conjecture in June 2026, and Wenlin Zhang independently sent him another (see the priority note); no priority is claimed.

Priority and subsequent correspondence

Added 30 October 2026. On 29 October 2026, following my correspondence with Professor Alexander Kovačec, he informed me that he had independently discovered a counterexample to his own conjecture in June 2026, documented in an unpublished preprint. He also mentioned that Wenlin Zhang had independently contacted him with another counterexample, apparently involving Lean. These developments were previously unknown to me, and the novelty searches carried out by my research agents had not found them. I acknowledge both earlier discoveries. I have not seen either counterexample, and it has not been established whether either is equivalent to the construction below. The construction, the proof and the formalization are unchanged.

Theorem 1. *Let* $a = (0, 1, i)$, $b = (-1 - i, 1, -1 + i)$ *and*

$$U = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Then U *is unitary,* $A = \operatorname{diag}(a)$ *and* $B = U \operatorname{diag}(b) U^*$ *are normal, and* $\det(A + B) = p = 2/3 + 2i/9$. *The boundary* Γ *of the straight-segment exterior of their full transposition graph is the Jordan polygon with consecutive vertices*

$$6 - 2i, \quad 2 + i, \quad 2i, \quad 1, \quad -i, \quad 2 - 2i.$$

There is a nonempty bounded connected component of $\mathbb{C} \setminus \Gamma$, *and* p *lies outside the closure of every bounded connected component. Nevertheless,* p *lies in the convex hull of the six permutation values.*

The definitions and proof follow, and Section 7 maps them to the Lean development. The gap is visible at height $2/9$: the left boundary has real coordinate $8/9$, while $\operatorname{Re} p = 2/3$. The argument must also identify that polygon as the boundary specified by the straight-segment exterior; choosing a cycle in the graph alone would not suffice.

1 The precise problem

For two lists of complex eigenvalues $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, write

$$D_a = \operatorname{diag}(a_1, \dots, a_n), \qquad D_b = \operatorname{diag}(b_1, \dots, b_n),$$

and define their determinant range

$$\Delta(a, b) = \{\det(D_a + U D_b U^*) : U \text{ is unitary}\}.$$

The permutation points are

$$z_\sigma = \prod_{j=1}^{n} (a_j + b_{\sigma(j)}), \qquad \sigma \in S_n.$$

The motivating enclosure proposal is attributed to Kovačec [1]. Its literal Definition 6, Conjecture 7, and current-version hypotheses have not been verified here. The following definitions are the exact model proved in this note. An external refutation is conditional on agreement with these definitions and quantifiers.

Draw a segment between two permutation points when the corresponding permutations differ by one transposition. Let G be the union of these segments. Call a point exterior if, for every $M > 0$, a straight segment of length M starting there avoids all edges of G. This is a visibility condition, distinct from an arbitrary path to infinity.

In formulas, for the closed real segment $[z, w] = \{(1-t)z + tw : 0 \le t \le 1\}$, the exterior is

$$E(G) = \{z : \forall M > 0 \ \exists w, \ |w - z| = M \text{ and } [z, w] \cap G = \varnothing\}.$$

The direction is permitted to depend on M. Write $\Gamma = \partial E(G)$.

The enclosure tested here places $\Delta(a, b)$ in the closure of a bounded complementary component when this boundary is a Jordan curve. The construction below violates this standalone enclosure, with the Jordan hypothesis verified for the specified spectra.

2 Exact normal matrices

Take

$$a = (0, 1, i), \qquad b = (-1 - i, 1, -1 + i),$$

and

$$U = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Let $\mathbf{J}$ denote the three-by-three all-ones matrix. Then $U = (2\mathbf{J} - 3I)/3$, and $\mathbf{J}^2 = 3\mathbf{J}$. Hence

$$U^2 = I, \qquad U^{\mathsf{T}} = U,$$

so U is real orthogonal and therefore complex unitary. Define $A = D_a$ and $B = UD_bU^*$. Both matrices are normal and have the required unitary spectral representations. Explicitly,

$$B = \frac{1}{9}\begin{pmatrix} -1+3i & -4+6i & 8 \\ -4+6i & -7 & -4-6i \\ 8 & -4-6i & -1-3i \end{pmatrix}.$$

The sum is

$$A + B = \frac{1}{9}\begin{pmatrix} -1+3i & -4+6i & 8 \\ -4+6i & 2 & -4-6i \\ 8 & -4-6i & -1+6i \end{pmatrix},$$

and direct determinant expansion gives

$$\det(A+B) = \frac{486+162i}{729} = \frac{2}{3} + \frac{2}{9}i = p.$$

For example, set $\alpha = -1+3i$, $\beta = -4+6i$, $\gamma = -4-6i$ and $\delta = -1+6i$. Expansion of the numerator matrix along its first row gives

$$\begin{aligned} 9^3 \det(A+B) &= \alpha(2\delta - \gamma^2) - \beta(\beta\delta - 8\gamma) + 8(\beta\gamma - 16) \\ &= \alpha(18 - 36i) - \beta(18i) + 288 \\ &= (90+90i) + (108+72i) + 288 = 486 + 162i. \end{aligned}$$

All quantities here are exact. No numerical eigenvalue calculation or floating-point tolerance enters the proof.

3 The permutation graph

Number the six permutations lexicographically, beginning at zero:

Index	Permutation	Permutation point
0	123	$v_0 = 6 - 2i$
1	132	$v_1 = 2$
2	213	$v_2 = 2 + i$
3	231	$v_3 = -i$
4	312	$v_4 = 2i$
5	321	$v_5 = 2 - 2i$

The complete list of unordered transposition edges is

$$01,\ 02,\ 05,\ 13,\ 14,\ 23,\ 24,\ 35,\ 45.$$

Each permutation has three transposition neighbors. Their index lists are, in order, $(1,2,5)$, $(0,3,4)$, $(0,3,4)$, $(1,2,5)$, $(1,2,5)$ and $(0,3,4)$. This exhausts the three transpositions at every permutation and yields exactly the nine unordered edges above.

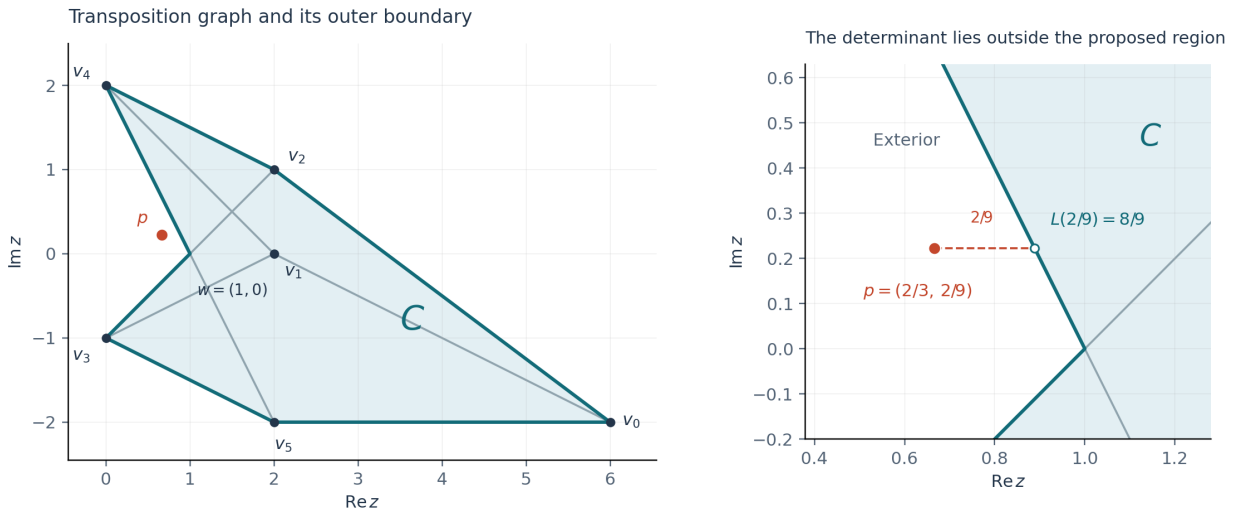


Figure 1: The complete transposition graph and its exterior boundary (left), with the indentation at $w = 1$ enlarged (right). The determinant p is a horizontal distance $2/9$ to the left of the polygon at height $2/9$. The exact formulas, rather than measurements from this drawing, prove the separation.

The segments $[v_2, v_3]$ and $[v_4, v_5]$ intersect at

$$w = 1.$$

Let C be the closed polygon whose vertices, in cyclic order, are

$$(6, -2), \quad (2, 1), \quad (0, 2), \quad (1, 0), \quad (0, -1), \quad (2, -2).$$

Every boundary segment belongs to the permutation graph: the successive sides are pieces of edges $02, 24, 45, 23, 35, 05$. Every graph edge lies in C.

4 Exact identification of the exterior and the Jordan boundary

The horizontal section of C at height y is

$$C \cap \{\operatorname{Im} z = y\} = \{x + iy : L(y) \le x \le R(y)\}, \qquad -2 \le y \le 2,$$

and is empty outside that height interval. The boundary functions are

$$L(y) = \begin{cases} -2y - 2, & -2 \le y \le -1, \\ y + 1, & -1 \le y \le 0, \\ 1 - y/2, & 0 \le y \le 2, \end{cases}$$

and

$$R(y) = \begin{cases} (10 - 4y)/3, & -2 \le y \le 1, \\ 4 - 2y, & 1 \le y \le 2. \end{cases}$$

The functions are continuous, and $R-L$ is affine on the intervals cut out by heights $-2,-1,0,1,2$. Its values at those heights are $4, 14/3, 7/3, 3/2, 0$, respectively. Hence $L < R$ below height 2, while $L(2) = R(2) = 0$. The set C is closed and bounded. Its interior is exactly

$$\{x+iy : -2 < y < 2,\ L(y) < x < R(y)\}.$$

Strict inequalities give interior points by continuity. Equality in either horizontal bound permits an arbitrarily small horizontal move outside C, and an extreme height permits a vertical move outside C. Thus the boundary is exactly the two graphs $x = L(y)$ and $x = R(y)$ on $[-2,2]$, joined by the bottom segment $[2-2i, 6-2i]$.

For completeness, every nonhorizontal graph edge has the following equation. The indicated height interval includes both endpoints.

Edge	Height interval	Real coordinate $x(y)$
01	$[-2,0]$	$2-2y$
02	$[-2,1]$	$(10-4y)/3$
13	$[-1,0]$	$2y+2$
14	$[0,2]$	$2-y$
23	$[-1,1]$	$y+1$
24	$[1,2]$	$4-2y$
35	$[-2,-1]$	$-2y-2$
45	$[-2,2]$	$1-y/2$

For each row, $L(y) \le x(y) \le R(y)$ on its interval: split at $-1, 0, 1$ and check the endpoint inequalities, since every difference is affine on each piece. The remaining edge 05 has $y = -2$ and $2 \le x \le 6$. This proves that the entire graph lies in C. Conversely, the right boundary consists of edges 02 and 24. The left boundary consists of edge 35, the part of edge 23 from $-i$ to 1, and the part of edge 45 from 1 to $2i$. Together with edge 05, these give the entire boundary. Hence ∂C lies in the graph.

For an explicit Jordan certificate, parameterize the six sides over $[0,6]$:

$$q(t) = \begin{cases} (6-4t) + i(-2+3t), & 0 \le t \le 1, \\ (4-2t) + it, & 1 \le t \le 2, \\ (t-2) + i(6-2t), & 2 \le t \le 3, \\ (4-t) + i(3-t), & 3 \le t \le 4, \\ (2t-8) + i(3-t), & 4 \le t \le 5, \\ (4t-18) - 2i, & 5 \le t \le 6. \end{cases}$$

The formulas agree at every joining point, so q is continuous and $q(0) = q(6) = 6-2i$. On $[0,2]$, the path follows the right graph $x = R(y)$ with height strictly increasing from -2 to 2. On $[2,5]$, it follows the left graph $x = L(y)$ with height strictly decreasing from 2 to -2. Since $L < R$ below the top, these arcs meet only at their common top endpoint. On $[5,6]$, the bottom path has height -2 and real coordinate strictly increasing from 2 to 6; it meets the other arcs only at its endpoints. Thus q is injective on $[0,6)$. The boundary description above shows that its image is precisely ∂C. This proves the Jordan condition directly.

The next step verifies the displayed exterior convention. Every point outside C has a straight escape ray. At heights between -2 and 2, a point to the left of $L(y)$ escapes horizontally left, and a point to the right of $R(y)$ escapes horizontally right. Points above or below the polygon escape vertically. Since the graph lies in C, these rays avoid every edge.

Conversely, a point on ∂C belongs to the graph, so a segment starting there already intersects the graph. A point in the interior of C cannot have an arbitrarily long graph-avoiding segment: choose a length M greater than the diameter of the compact set C. Every endpoint at distance M from the interior point is outside C. A segment joining the two points must meet ∂C: otherwise its connected image would meet both of the disjoint open sets $\operatorname{int} C$ and $\mathbb{C} \setminus C$ while lying in their union. Since ∂C is contained in the graph, no such segment avoids the graph.

Consequently the displayed exterior is **exactly** $\mathbb{C} \setminus C$, and its boundary is the verified Jordan curve ∂C.

5 Failure of the defined enclosure

At the determinant's height,

$$\operatorname{Im} p = \frac{2}{9}, \qquad L\left(\frac{2}{9}\right) = \frac{8}{9},$$

whereas

$$\operatorname{Re} p = \frac{2}{3} < \frac{8}{9}.$$

Thus $p \notin C$, with an exact horizontal gap of $2/9$ from the left boundary. Its entire left horizontal ray stays outside C, and hence outside the permutation graph.

For completeness, the formal proof also addresses the bounded complementary component directly. Every point outside C has an unbounded connected escape ray lying outside its frontier. The connected component of $(\partial C)^c$ containing that point is therefore unbounded. It follows that every bounded connected component of $(\partial C)^c$ lies inside C; since C is closed, its closure also lies in C. Therefore

$$p \notin \overline{K} \quad \text{for every bounded connected component } K \text{ of } (\partial C)^c.$$

There is a bounded nonempty component: $2 + 0i$ is an interior point because

$$-2 < 0 < 2, \qquad L(0) = 1 < 2 < R(0) = \frac{10}{3}.$$

The complement of the frontier is the disjoint union of the open sets $\operatorname{int} C$ and $\mathbb{C} \setminus C$. Its connected component containing 2 meets the first, so connectedness prevents it from meeting the second. The component is therefore nonempty and contained in the bounded set C. The counterexample consequently satisfies the displayed existence and Jordan hypotheses, and defeats every possible bounded-component enclosure. No unproved general Jordan curve theorem or uniqueness assertion is used in the formal proof.

This proves failure of the stated polygon enclosure, with all of its geometric hypotheses satisfied.

6 Why this does not contradict the classical conjecture

The determinant has the explicit convex representation

$$p = \frac{1}{9}v_0 + \frac{4}{9}v_3 + \frac{4}{9}v_4.$$

The three weights are nonnegative and sum to one. Thus p belongs to the ordinary convex hull of the permutation points. The three points used here correspond to the even permutations

123, 231, 312; no pair of them is joined by a single-transposition edge. Their convex combinations need not stay within the smaller polygon defined here.

The exact identity is checked by the following Lean declaration:

`KovacecResolution.ordinary_convex_combination`

7 Formal verification and scope

The proof above establishes Theorem 1 without a computer calculation. Its formal counterpart consists of eight proof modules and a root import file. The matrix certificate verifies the actual matrices, unitarity, normality, determinant, all six permutations and all nine transposition edges. The geometric modules define the boundary functions, prove the exact frontier and loop image, and verify injectivity. The exterior and component modules prove equality with the straight-segment exterior and exclusion from all bounded component closures.

The endpoint `KovacecResolution.completed_counterexample` includes the two normality identities, exact determinant, Jordan condition, existence of a nonempty bounded component, and exclusion from every bounded component closure. The package-defined predicate `Conjecture7` asks for a bounded component closure containing the determinant under the displayed hypotheses. The theorem `conjecture7_false` refutes that formal predicate. These names do not authenticate an external Conjecture 7 or its definitions. No general Jordan-curve uniqueness theorem is needed. The separate theorem `ordinary_convex_combination` verifies the convex identity.

The checked endpoint dependencies are exactly `propext`, `Classical.choice` and `Quot.sound`. The formalization map identifies the declarations and reproduction commands. Formal verification checks these stated objects and assumptions; it does not establish novelty, external source identity, or publication priority.

The result concerns the stated polygon enclosure. It does not refute the Marcus–de Oliveira convex-hull conjecture or resolve that conjecture in general. Its scope is an exact obstruction in dimension three, with a real orthogonal mixing matrix and the full geometric hypotheses verified.

8 Research process and AI disclosure

This work is fully autonomous agentic research. Automated systems produced the research, proofs, formalization, and earlier versions of this manuscript. Opus 5.5 by Anthropic used with Claude Code revised the presentation of the manuscript on 1 November 2026: it rewrote the front matter and abstract and consolidated this disclosure. The revision does not change the mathematics. Dev D. Goyal is responsible for publication. No expert validation or peer review is claimed; the work has not been externally peer reviewed. The formal scope and its limits are stated in Section 7, and the earlier counterexamples in the priority note.

References

[1] Alexander Kovačec. A conjecture more precise and stronger than the one by Marcus and de Oliveira. Technical Report 26-11, Departamento de Matemática, Universidade de Coimbra, April 2026. Unverified source-context metadata. Version 1, 7 April 2026. https://www.mat.uc.pt/preprints/ps/p2611.pdf.